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Stabilization of cyclic processes by slowly varying forcing

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COLLECTIONS

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ABSTRACT

We introduce a new mathematical framework for the qualitative analysis of dynamical stability, designed particularly for finite-time processes subject to slow-timescale external influences. In particular, our approach is to treat finite-time dynamical systems in terms of a slow–fast formalism in which the slow time only exists in a bounded interval, and consider stability in the singular limit. Applying this to one-dimensional phase dynamics, we provide stability definitions somewhat analogous to the classical infinite-time definitions associated with Aleksandr Lyapunov. With this, we mathematically formalize and generalize a phase-stabilization phenomenon previously described in the physics literature for which the classical stability definitions are inapplicable and instead our new framework is required.

An important phenomenon in nonlinear science is the stabilization of processes by external forces. This phenomenon can manifest as the mutual synchronization of an array of processes evolving under the same law but starting at different initial states, where the synchronization is due to the processes being subject to a common external forcing. Such stabilization is traditionally analyzed in terms of *Lyapunov exponents* and other related long-time-asymptotic conceptions of dynamical stability; these can be applied to both deterministic and stochastic dynamical systems that are well-defined over infinite time. References 1 and 2 present a stabilization phenomenon in which the kind of external forcing considered is a slow-timescale process; our present paper concerns mathematically rigorous formulation, proof, and generalization of this stabilization phenomenon. We will see that the traditional approaches to theoretical stability analysis such as Lyapunov exponents and other infinite-time approaches cannot generally be applied to describe the stabilization phenomenon of Refs. 1 and 2. In fact, the critical forcing strength required to induce stability is generally dependent on the time-interval over which the system is being considered. This means that the phenomenon should be understood in terms of *finite-time* stability, in contrast to other stabilization phenomena such as the well-known case of stabilization by stationary noise. Our work represents a significant contribution to the important and growing field of finite-time dynamical systems (FTDSs) and finite-time stability, whose necessity in the study of various nonlinear systems, such as climate and biological systems, is increasingly being recognized.

I. INTRODUCTION

For either an *autonomous* dynamical system

$$\dot{x}(t) = F(x(t)), \quad t \in \mathbb{R}, \tag{1}$$

with a time-independent vector field $F\colon X \to TX$, or a *nonautonomous*[3–5] dynamical system

$$\dot{x}(t) = F(x(t), t), \quad t \in \mathbb{R},$$

with a time-dependent vector field $F\colon X \times \mathbb{R} \to TX$, dynamics is traditionally analyzed in terms of behavior as $t \to \pm\infty$. For example, attractors, repellers, asymptotic stability, and Lyapunov exponents are typically defined in terms of such long-time-asymptotic behavior. From a practical perspective, this approach is based on the assumption that one is interested in the system's behavior *after a sufficiently long time that "transients have decayed."*

However, in more recent decades, the study of "finite-time dynamical systems" (FTDSs) has grown in popularity;[6–14] these are dynamical systems in which *time is constrained to a compact interval*, i.e.,

$$\dot{x}(t) = F(x(t), t), \quad t \in [0, \mathfrak{T}]$$

for some time-dependent vector field $F\colon X \times [0, \mathfrak{T}] \to TX$. As described in Secs. 1.1 and 6 of Ref. 11, situations where the FTDS framework is appropriate include

(i) when one is interested in behavior on a shorter timescale than that required for a given model's long-time-asymptotic dynamical picture to emerge, or
(ii) when the temporal variations of the external forcing over finite timescales of interest are "*unextendible,*" meaning that they do not (or are not know to) follow any particular indefinite-time deterministic or stochastic model, and so there is no such thing as "long-time-asymptotic" behavior.

Now sometimes, the temporal variations of the external forcing may be relatively slow, and this slowness may play an important role in shaping the consequent dynamical behavior.[15] In this paper:

- We will introduce (immediately below, and further in Sec. III) a framework for investigating the role of slow variation of external forcing on the consequent dynamical behavior, specifically designed to be applicable to "unextendible" temporal variations defined over compact time-intervals. (The need for such a framework will be further illustrated in Sec. II B.)
- We will apply this framework to provide a rigorous mathematical description of a certain one-dimensional phase-stabilization phenomenon described in the physics literature, in Refs. 1 and 2.

The framework we introduce is a slow–fast setup in which *the slow time is constrained to a compact interval* (without loss of generality, the unit interval). That is, we consider systems of the form

$$\dot{x}(t) = F(x(t), \varepsilon t), \quad t \in [0, \tfrac{1}{\varepsilon}] \tag{2}$$

for some $[0, 1]$-parameterized vector field,

$$F\colon X \times [0, 1] \to TX.$$

While traditional mathematical definitions of stability and Lyapunov exponents are formulated in terms of the $t \to \infty$ behavior, we will formulate analogous definitions in terms of the $\varepsilon \to 0$ behavior of (2). While a classical $t \to \infty$ limit represents "non-*transient* dynamical phenomena," the $\varepsilon \to 0$ limit represents "non-*rate-induced*[16] dynamical phenomena." Here, ε represents the "timescale separation" between the slower timescale of the external forcing and the faster timescale of the internal dynamics; this setup can equivalently take the form

$$\dot{x}(t) = \tfrac{1}{\varepsilon} F(x(t), t), \quad t \in [0, 1]. \tag{3}$$

Now, in investigating the stability properties of (2) and (3), this paper focuses on the natural "starting case" to consider, namely, dynamics on a one-dimensional phase space (specifically, the circle $X = \mathbb{S}^1$). This already provides significant non-trivialities to deal with (see Sec. II C), especially the *canard* phenomenon[17] of multiple-timescale dynamics where solutions spend a long time traveling along an unstable portion of the slow manifold. Our results in this paper will both formalize and considerably generalize the core findings of Refs. 1 and 2 regarding stabilization of the one-dimensional Adler equation by slow-timescale additive forcing. The two main steps that will be involved are

(a) showing that "generic" time-dependent vector fields $F\colon \mathbb{S}^1 \times [0, 1] \to T\mathbb{S}^1$ admit an adiabatically traceable moving sink analogous to the moving sink of the slowly varying Adler equation (Theorem 14); and
(b) making mathematically rigorous the heuristic stability arguments used in Refs. 1 and 2 for the Adler equation (Theorem 24), which we mainly achieve by adapting methods that were used in Ref. 17 for investigating periodic orbits of an Adler equation subject to low-frequency sinusoidal forcing.

We will also analyze the generic behavior of the transition to stability from neutral stability induced by slow-timescale forcing, and its contrast to analogous classical saddle-node bifurcations (see Secs. III G, IV D, and IV E 3). Let us also mention that *fast*-timescale forcing of the Adler equation as a FTDS has been considered in Theorem 6.3 of Ref. 18, as part of a study of phase-synchronization transitions.

From a physical perspective, dynamical systems on $\mathbb{S}^1$ can be used to describe the phase of a cyclic process via *phase reduction*, which has been established for systems subject to slow-timescale forcing in Refs. 19–21. The study of one-dimensional phase dynamics has particular relevance for various *biological* oscillatory processes or coupled pairs of oscillatory processes;[1,2,18,21–23] and indeed, in such contexts, it is especially important that temporal variations in external forcing are "unextendible" in the sense described above, otherwise, the organism cannot properly interact with the outside world and will quickly die.

However, we repeat that the rationale behind this paper is not simply to obtain results about one-dimensional phase dynamics. We hope through our treatment of the one-dimensional case of (2) to help lay the foundations for the development of more general methodology for treating FTDSs involving multiple timescales. The need for theory of higher-dimensional dynamics within the framework of Eqs. (2) and (3) is already evidenced in Sec. III of Ref. 2, where numerics indicate that the phenomenon of stabilization of solutions by slowly varying forcing can also occur in higher-dimensional systems, including chaotic systems. Additionally, Ref. 24 shows that the same mechanism of stabilization described in Refs. 1 and 2 for one-dimensional dynamics can also induce local stability of synchronous solutions in a driven Kuramoto network of arbitrary size.

So, we anticipate that the line of inquiry initiated by this paper may find utility in the investigation of various complex systems such as biological oscillator networks (where our earlier comment about the necessity of "unextendible" temporal variations still applies) and climate systems subject to natural or anthropogenic forcing. Furthermore, we believe that mathematical theory of multiple-timescale FTDSs can provide deeper insight into the nature and application of time-series analysis tools designed for studying time-localized dynamics, such as those described in Refs. 5, 25, and 26.

The paper is organized as follows.

In Sec. II, we describe the stabilization of the Adler equation presented in Refs. 1 and 2, its physical motivations, and its contrast to the traditional cases of stabilization by constant forcing and by noise. We illustrate how the stability and neutral stability observed in Refs. 1 and 2 cannot generally be formulated in terms of the traditional definitions and quantification of stability due to Aleksandr Lyapunov, such as asymptotic stability and asymptotic Lyapunov exponents. We then outline the primary mathematical non-trivialities for our main task of providing a mathematically rigorous formulation and proof of the stabilization phenomenon.

In Sec. III, we first introduce our concepts of stability for systems of the form (2) defined on the circle $X = \mathbb{S}^1$. Then, we present other definitions necessary for the formulation of our main results (such as curves of the stable and unstable slow manifolds and tracking of such curves) and present our main results that mathematically formalize and generalize the basic argument of Refs. 1 and 2. We also outline our main methods of proof and their relation to the methods in Ref. 17. Finally, we give a result regarding the generic scenario of a transition between stability and neutral stability within our framework, highlighting, in particular, its signature characteristic of approximately *linear* growth of the stability exponent on the stable side of the transition (in contrast to the *square-root* growth in classical saddle-node bifurcations).

In Sec. IV, we carry out a comparison of our new framework of stability analysis with the classical long-time-asymptotic framework, particularly through the example of low-frequency-periodically forced Adler equations.

In Sec. V, we prove the results in Sec. III.

In Sec. VI, we summarize and conclude.

In Appendix A, for the sake of completeness, we obtain a much more refined version of one of the results (Corollary 45) needed within the proof of Theorem 24(C).

In Appendix B, we state the numerical procedures used for all the numerics in this paper.

A. Note

A relatively non-technical presentation of some of the discussions in this paper (in particular, in Secs. II A, II B, IV B 1, IV B 3, IV C, and IV E 1) has been published in Ref. 27 (which is itself the sequel to an exposition of phase synchronization by time-dependent driving given in Ref. 28).

II. STABILIZATION OF ADLER EQUATIONS BY SLOW-TIMESCALE FORCING

In Sec. II A, we first give preliminaries regarding the general concept of stabilization by external forcing and two basic mechanisms of stabilization of the Adler equation; then, we describe the stabilization by slow-timescale forcing studied in Refs. 1 and 2. In Sec. II B, we will demonstrate an example of this latter mechanism of stabilization, where we use a slow-timescale forcing that is "unextendible" in the sense described in Sec. I. In Sec. II C, we will outline the main non-trivialities in obtaining a mathematically rigorous formulation and proof of this stabilization phenomenon.

A. Stabilization by external forcing

It is well-known that trajectories of an autonomous dynamical system (1) can often be "stabilized" by the introduction of external forcing, e.g., an additive forcing $\xi(t)$ giving rise to a nonautonomous dynamical system of the form

$$\dot{x} = F(x) + \xi(t). \tag{4}$$

This "stabilization" can be defined in terms of synchronization between solutions starting at different initial conditions:[29] namely, it refers to the scenario that for an array of distinct initial conditions $x_1^*, \ldots, x_n^*$, the solutions of (1) starting at these initial conditions remain clearly distinct at all times but the solutions of (4) starting at these same initial conditions become synchronized with each other. Note that this synchronization is not induced by any "direct" coupling between the solutions but rather by the "indirect coupling" of being simultaneously subject to the same external driving $\xi(t)$. A natural question then is what kinds of forcing function $\xi(t)$ will give rise to such synchronization.

As a prototype: on the circle $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$, all solutions of the autonomous *Adler equation*

$$\dot{\theta}(t) = -a\sin(\theta(t)) + k, \tag{5}$$

with constant parameters $k > a > 0$, move strictly periodically round the circle with common period $\frac{2\pi}{\sqrt{k^2-a^2}}$; and so, in particular, the solutions of any two distinct initial conditions do not synchronize with each other. We now consider what types of external forcing can cause the solutions of different initial conditions to become synchronized with each other.

1. Stabilization by constant forcing

One crude way to stabilize system (5) is to apply a perpetually constant external forcing ξ such that the "effective new k-value" $k_{\text{new}} = k + \xi$ lies in $(-a, a)$. In this case, all solutions now converge to the point $y_{k_{\text{new}}} := \arcsin\left(\frac{k_{\text{new}}}{a}\right)$ apart from one single exceptional solution that remains fixed at the point $\pi - y_{k_{\text{new}}}$; and thus, the phase differences between these solutions trivially tend to 0 as $t \to \infty$. This mutual synchronization of solutions takes place at an exponential rate with exponent $-\sqrt{a^2 - k_{\text{new}}^2}$, called the *Lyapunov exponent* associated with the fixed point $y_{k_{\text{new}}}$. [This exponent is obtained as the derivative of $-a\sin(\cdot) + k_{\text{new}}$ at $y_{k_{\text{new}}}$.]

Note that if one were instead to take the constant ξ sufficiently close to zero, then the Adler equation would not be stabilized: solutions would move strictly periodically round the circle with common period $\frac{2\pi}{\sqrt{k_{\text{new}}^2-a^2}}$. The bifurcation that takes place as ξ crosses the critical threshold $a - k$ is a *saddle-node bifurcation*, namely, the creation of a stable and an unstable equilibrium point.

Now, from a practical point of view, this crude approach to the removal of asynchrony among solutions through simply forcing them perpetually to remain at a fixed point may (depending on context) be problematic due to the inability of solutions to move freely throughout the phase space. It may also (again depending on context) be rather energy-inefficient to keep this forcing perpetually maintained.

2. Stabilization by noise

One kind of forcing that is often known to induce stability in dynamical systems is *noise*.[29–31] Noise-induced stability is very common in general but is especially typical for one-dimensional phase-oscillator dynamics, where, for example, memoryless stationary noise is guaranteed to lead to mutual synchronization of solutions under weak conditions.[32,33]

As an example, the addition of stationary Gaussian white noise of intensity $A > 0$ to the Adler equation (5) will stabilize the Adler equation, regardless of how small A is.[29,34] To be more precise: if we fix the values of k, a and A, almost every realization $W\colon [0,\infty) \to \mathbb{R}$ of a Wiener process has the property that all solutions of the corresponding nonautonomous dynamical system

$$d\theta(t) = (-a\sin(\theta(t)) + k)\,dt + A\,dW(t) \tag{6}$$

synchronize with each other in the limit as $t \to \infty$ (i.e., their phase differences tend to 0), apart from one solution starting at a single exceptional initial condition. Moreover, this synchronization of solutions comes with a well-defined Lyapunov exponent (i.e., long-time-asymptotic exponential rate of decay of phase differences), which is exactly the same value across almost all realizations W of the Wiener process.

The mechanism behind this exponential synchronization of solutions is that the infinite time-horizon allows for statistical large-number laws to come into play, whereby random events in the noise which bring the different solutions toward those regions on the circle where they are pushed closer together are guaranteed to occur with sufficient long-time-asymptotic regularity. Note, in particular, that if the noise intensity A is very small, then it will take a very long time for the synchrony among different solutions to begin to emerge.

3. The stabilization phenomenon of Refs. 1 and 2

In contrast to stationary white noise, a very different type of forcing that one can consider is deterministic[35] slowly varying forcing. In particular, in the physics literature, Refs. 1 and 2 considered equations of the form

$$\dot{\theta}(t) = -a\sin(\theta(t)) + G(t), \tag{7}$$

with $G(t)$ having slow dependence on t. In both of these papers, the variable $\theta(t)$ in (7) is regarded as representing the phase difference between a unidirectionally coupled pair of phase oscillators (where the driving oscillator has time-variable frequency), so that synchronization between the solutions of (7) starting at different initial conditions corresponds to stabilization of the driven oscillator by the driving oscillator. Physical motivations in Ref. 1 include the van der Pol oscillator subject to aperiodic external driving [where a phase-reduction approximation brings the system to the form (7)], as well as neuron voltage spiking dynamics, where stabilization by external driving is known to occur.[36–38] One physical motivation behind Ref. 2 was to develop upon some of the main findings of Refs. 22 and 23 for cardiorespiratory phase synchronization.

The foundational point of Refs. 1 and 2 is essentially the following argument:

- If there are times t at which the slowly varying function $G(t)$ lies in the interval $(-a, a)$, then during such times, the solutions of (7) will cluster together into an increasingly tight cluster around the "slowly moving sink" $y_{G(t)} = \arcsin\left(\frac{G(t)}{a}\right)$.
- By contrast, while $G(t)$ is outside the interval $[-a, a]$, solutions move unboundedly round the circle without either synchronizing with each other or becoming desynchronized from each other.
- Thus, overall, we have mutual synchronization of solutions if there are times when $G(t)$ enters the interval $(-a, a)$, but not if $G(t)$ stays outside the interval $[-a, a]$. (We will illustrate this further in Sec. II B.)

As in Ref. 1, we can derive an approximation Λ of the average exponential rate of mutual synchronization of solutions, by adiabatically following the Lyapunov exponent of the stable fixed point $y_{G(t)}$ of the time-frozen vector field $-a\sin(\cdot) + G(t)$ when it exists. More precisely, over a time-interval $[0, T]$, the approximate average exponential rate of synchronization (taken as negative when such synchronization does occur) is given by

$$\Lambda = -\frac{1}{T}\int_{\{s\in[0,T]:|G(s)|<a\}} \sqrt{a^2 - G(t)^2}\,dt. \tag{8}$$

Note that Λ is 0 if $G(t)$ never enters the interval $(-a, a)$.

Now, as in Ref. 2, we can consider $G(t)$ of the form $G(t) = k + Ag(t)$ for some $k > a$, some function $g(t)$ taking both positive and negative values, and some parameter $A \geq 0$; so (7) becomes

$$\dot{\theta}(t) = -a\sin(\theta(t)) + k + Ag(t). \tag{9}$$

This represents the addition of an external forcing $+Ag(t)$ to the autonomous system (5). Parameter A is somewhat analogous to the noise intensity parameter in (6), except that we are now considering slow-timescale deterministic variation rather than fast-timescale stochastic variation. Fixing the function g, if A is sufficiently large, then there will be times t at which $k + Ag(t)$ enters the interval $(-a, a)$, and so the added forcing $+Ag(t)$ will induce stabilization.

References 1 and 2 also provide strong numerical support for the above description of the dynamics of (7) and (9). The numerics in Ref. 2 use low-frequency sinusoidal forcing (see also Secs. IV C and IV E 2 of this paper), and Ref. 1 considers various other forms of forcing. In Sec. II B, we will go on to provide further numerical support using a form of forcing deliberately designed to be "unextendible" in the sense of point (ii) in the Introduction.

B. An illustrative example of stabilization by "unextendible" slow-timescale forcing

To illustrate the need for a FTDS framework in order to be able to formulate mathematically the stabilization phenomenon described in Refs. 1 and 2, we consider Eq. (9) with $g(t)$ being a function that is only meaningfully defined on a compact time-interval.

One way in which such a function $g(t)$ could arise is as an empirically recorded time-series:[11] if one defines a nonautonomous dynamical system with reference to an actual empirically measured signal for the external forcing, then the finiteness of the signal duration means that the system is inherently a FTDS, for which traditional infinite-time dynamical systems theory cannot be used as a framework for dynamics analysis. Forcing a fast-timescale Adler equation by a finite-duration empirical signal measured from a relatively slow-timescale physical process will give results in line with the picture described in Sec. II A 3.

But here, we will construct $g(t)$ numerically. Specifically, we will generate a sample realization of a *Brownian bridge*—this is a finite-time stochastic process, which represents the inhomogeneities of a large sample of random times selected with uniform probability from a pre-specified compact time-interval. Then, as in Ref. 1, Sec. III B, we will pass the result through a low-pass filter to obtain our slowly time-dependent forcing. Our choice of using a sample realization of a Brownian bridge is not motivated by any specific physical application but is simply for illustrative purposes as a good example of "unextendible" temporal variations.

1. The forcing and the system parameters

Let us first briefly recall the definition of a Brownian bridge, and its connection to Brownian motion. A *Brownian bridge of parameter* $\sigma > 0$ *on a time-interval* $[0, \mathfrak{T}]$ is a finite-time stochastic process $(B_t)_{t\in[0,\mathfrak{T}]}$ approximated by taking a very large simple random sample of times $S = \{t_1, \ldots, t_N\}$ from the uniform distribution on $[0, \mathfrak{T}]$ and then setting

$$B_t = \sigma\sqrt{N\mathfrak{T}}\,\mathscr{E}(t),$$

where $\mathscr{E}(\cdot)$ is the "signed deviation from homogeneity" of the random sample S, that is,

$$\mathscr{E}(t) = \frac{\#(S \cap [0, t])}{N} - \frac{t}{\mathfrak{T}}.$$

This definition is made rigorous by *Donsker's theorem*.[39] Now, although a *Brownian motion* $(M_t)_{t\geq 0}$ is an indefinite-time process, one can construct up to time $\mathfrak{T}$ a zero-drift Brownian motion M_t of diffusion parameter σ by taking a random variable $Y \sim \mathscr{N}(0, \frac{\sigma^2}{\mathfrak{T}})$ that is independent of the Brownian bridge $(B_t)_{t\in[0,\mathfrak{T}]}$ and then letting

$$M_t = tY + B_t. \tag{10}$$

For each $t \in [0, \mathfrak{T}]$, the variance of B_t is given by

$$\mathrm{var}(B_t) = \mathrm{var}(M_t) - \mathrm{var}(tY) = \sigma^2 t(1 - \tfrac{t}{\mathfrak{T}}).$$

This quadratic expression with respect to t is non-negative on $[0, \mathfrak{T}]$ but becomes negative when t is outside of $[0, \mathfrak{T}]$. Thus, there is no natural way to extend a Brownian bridge on $[0, \mathfrak{T}]$ beyond time $\mathfrak{T}$. [And indeed, there is no way to extend the construction (10) beyond time $\mathfrak{T}$, as the variance of tY then exceeds the variance of M_t.]

Now, in our case, we define $g(t)$ on a compact interval $[0, \mathfrak{T}]$ with $\mathfrak{T} = 2\pi \times 10^5$ s; we constructed $g(t)$ by first generating a sample realization of a Brownian bridge on $[0, \mathfrak{T}]$ of parameter $\sigma = \frac{1}{\sqrt{\mathfrak{T}}} \approx 1.3 \times 10^{-3}\,\mathrm{s}^{-\frac{1}{2}}$ and then passing the result through a fifth order Butterworth low-pass filter with cut-off frequency $1/(2\pi \times 10^3) \approx 1.6 \times 10^{-4}$ Hz. The resulting function is shown is Fig. 1. Further details of this construction and of all numerics in this paper are given in Appendix B. For other parameters in Eq. (9), we take $a = \frac{1}{3}$ rad/s and $k = 1$ rad/s, and we consider how the dynamics depends on the parameter $A \geq 0$. Due to the low-pass filter, $g(t)$ has very slow dependence on t compared to the timescale of the "internal dynamics" of the system as represented by Eq. (5), whose solutions have a frequency of $\frac{\sqrt{k^2-a^2}}{2\pi} \approx 0.15$ Hz.

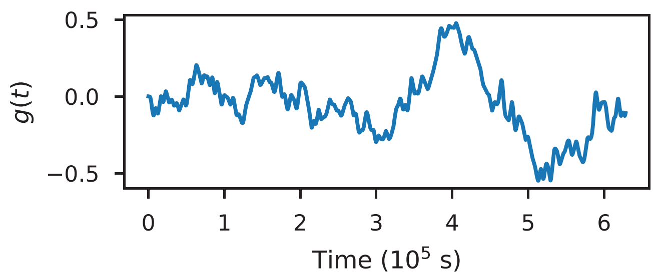

FIG. 1. Graph of $g(t)$ obtained by passing a sample realization of a Brownian bridge on $[0\,\mathrm{s}, 2\pi \times 10^5\,\mathrm{s}]$ through a low-pass filter of cut-off frequency $1/(2\pi \times 10^3)$ Hz. Reproduced with permission from Newman *et al.*, *Physics of Biological Oscillators*, edited by A. Stefanovska and P. V. E. McClintock (Springer, 2021), Chap. 7, pp. 111–129. Copyright 2021 Springer Nature Switzerland AG.

2. The stabilization phenomenon

Considering (9) over the whole time-interval $[0, \mathfrak{T}]$, the argument described in Sec. II A 3 tells us that if A is less than $A_* := \frac{a-k}{\min_{t\in[0,\mathfrak{T}]} g(t)}$ then the solutions of (9) do not synchronize with each other, but if A is greater than A_* then the solutions of (9) do synchronize with each other. This is exactly what we observe in numerics: Fig. 2(b) shows a "numerical bifurcation diagram" of (9) where for each A-value, the trajectories at time $\mathfrak{T}$ of 50 evenly spaced initial conditions are shown; and in dashed black is marked the value of A_*.

- For $A < A_*$, we see the trajectories distributed fairly evenly distributed throughout the circle, just as we would have seen for the unperturbed system (5). We refer to this lack of clear mutual attraction or mutual repulsion between the trajectories of different initial conditions as *"neutral stability."* Now, in the setting of infinite-time dynamical systems, one can give a rigorous definition of "neutral stability" in terms of the traditional framework of long-time-asymptotic dynamics, such as we do in Sec. IV A. (The definition there implies, in particular, that all solutions are Lyapunov stable but not asymptotically stable, and have a Lyapunov exponent of zero.)
- For $A > A_*$, we see that the trajectories are "stabilized," i.e., they have become mutually synchronized with each other, such that they appear like a single point in Fig. 2(b).

Thus, a clear transition from "neutral stability" to "stability" takes place at $A = A_*$. Figure 2(c) shows the "numerical reverse-time bifurcation diagram," where 50 evenly spaced initial conditions are

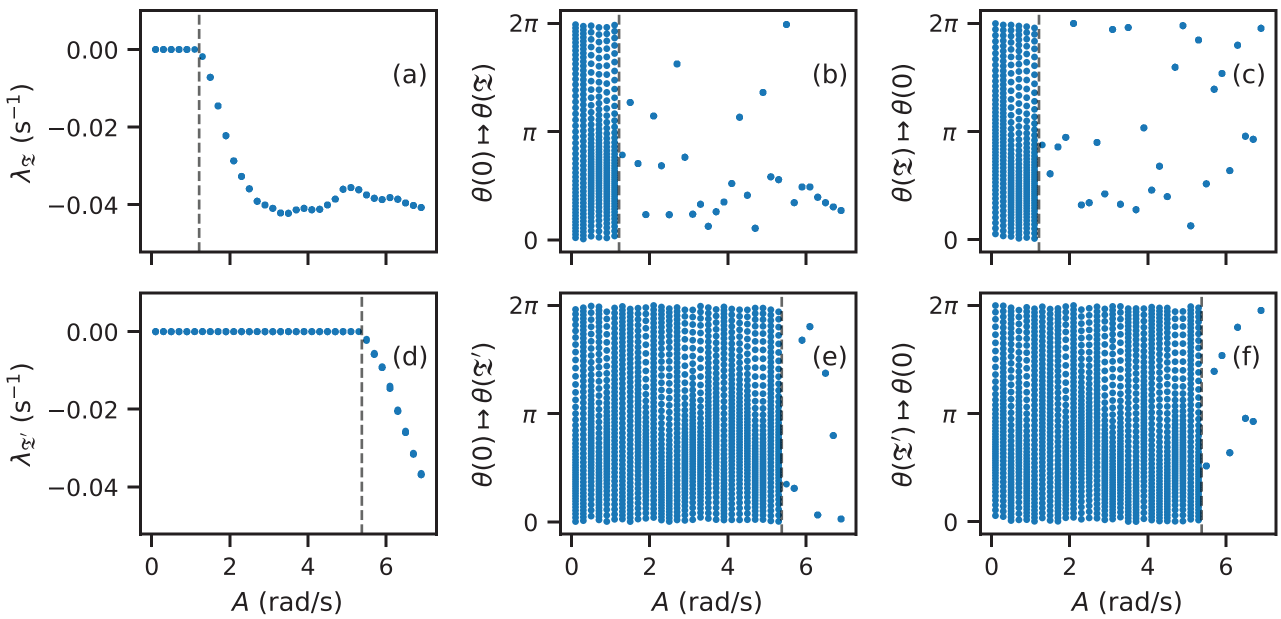

FIG. 2. Dynamics of (9) with g as in Fig. 1, with varying A, over the time-interval $[0\,\text{s}, 2\pi \times 10^5\,\text{s}]$ of duration $\mathfrak{T} = 2\pi \times 10^5$ s in (a)–(c), and over the shorter time-interval $[0\,\text{s}, \pi \times 10^4\,\text{s}]$ of duration $\mathfrak{T}' = \pi \times 10^4$ s in (d)–(f). Other parameters are $a = \frac{1}{3}$ rad/s and $k = 1$ rad/s. In (a), (b), (d), and (e), for each A-value, results for the evolution $\theta(t)$ of 50 equally spaced initial conditions $\theta(0) = \frac{2\pi i}{50}$, $i = 0, \ldots, 49$, are shown: (a) and (d) show the finite-time Lyapunov exponents λ_T, as defined by (11), for these trajectories, with $T = \mathfrak{T}$ in (a) and with $T = \mathfrak{T}'$ in (d); (b) and (e) show the positions $\theta(T)$ of these trajectories at time $T = \mathfrak{T}$ in (b) and $T = \mathfrak{T}'$ in (e). In (c) and (f), for each A-value, the positions of $\theta(0)$ for the 50 trajectories ending at the points $\theta(T) = \frac{2\pi i}{50}$, $i = 0, \ldots, 49$, are shown, with $T = \mathfrak{T}$ in (c) and $T = \mathfrak{T}'$ in (f). In (a)–(c), the value $A_* := \frac{a-k}{\min_{t\in[0,\mathfrak{T}]} g(t)}$ is marked in dashed black. In (d)–(f), the value $A'_* := \frac{a-k}{\min_{t\in[0,\mathfrak{T}']} g(t)}$ is marked in dashed black. Reproduced (with minor modification) with permission from Newman *et al.*, *Physics of Biological Oscillators*, edited by A. Stefanovska and P. V. E. McClintock (Springer, 2021), Chap. 7, pp. 111–129. Copyright 2021 Springer Nature Switzerland AG.

run in backward time under (9) from time $\mathfrak{T}$ to time 0; we see from this that when A rises above A_*, the forward-time solutions of (9) are being mutually repelled from a very small region [appearing like a single point in Fig. 2(c)]. So, overall, the picture indicated by plots (b) and (c) of Fig. 2 is rather resemblant of a saddle-node bifurcation in classical autonomous dynamical systems theory, where a lack of equilibria bifurcates into a repelling (unstable) and an attracting (stable) equilibrium. However, the system is a finite-time nonautonomous dynamical system: mathematically rigorous formulation of the "bifurcation" seen in plots (b) and (c) cannot be in terms of traditional "$t \to \infty$" considerations by which concepts such as "stable" and "unstable" equilibria are defined.

3. Analysis via finite-time Lyapunov exponents

Local stability of solutions of a dynamical system can be quantified in terms of *Lyapunov exponents*. Traditional infinite-time dynamical systems theory tends to focus on long-time-asymptotic Lyapunov exponents, simply referred to as "Lyapunov exponents" without a qualifier. These Lyapunov exponents are defined as the long-time-asymptotic limit of *finite-time Lyapunov exponents* (FTLEs). In some contexts, one can also study other long-time-asymptotic features of the set of the FTLEs, such as given by large deviation theory[29] or the dichotomy spectrum.[40] Now, in our finite-time system, we cannot apply any long-time-asymptotic concepts; but we can still apply the concept of FTLEs within the bounded time-interval $[0, \mathfrak{T}]$.

The finite-time Lyapunov exponent for a solution $\theta(\cdot)$ of (9) over a time-window $[0, T]$ is calculated by averaging over time t the derivative of the time-frozen vector field $\theta \mapsto -a\sin(\theta) + k + Ag(t)$ at $\theta(t)$, i.e.,

$$\lambda_T = \frac{1}{T}\int_0^T -a\cos(\theta(t))\,dt. \tag{11}$$

Working over the whole time-interval $[0, \mathfrak{T}]$, Fig. 2(a) shows the values of $\lambda_{\mathfrak{T}}$ for the trajectories of 50 initial conditions. For each A-value, we see that the 50 trajectories share indistinguishably the same FTLE value as each other (appearing like a single point), being indistinguishable from 0 for $A < A_*$ and clearly negative for $A > A_*$. Again, this suggests a transition from neutral stability to stability as A crosses A_*.

4. Restricting to a time-subinterval

Now, for further illustration, let us consider the dynamics of (9) not over the whole time-interval, but rather over a subinterval $[0, \mathfrak{T}']$ with $\mathfrak{T}' = \pi \times 10^4$ s. Figure 2(e) shows the numerical bifurcation diagram for simulation only up to time $\mathfrak{T}'$, and likewise Fig. 2(f) shows the numerical reverse-time bifurcation when running initial conditions backward in time from time $\mathfrak{T}'$ to time 0. Here, we see the saddle-node-like "bifurcation" occurring at $A'_* := \frac{a-k}{\min_{t\in[0,\mathfrak{T}']} g(t)}$, again exactly in accordance with the reasoning of Sec. II A 3. In Fig. 2(d) are shown FTLEs of trajectories over the time-window $[0, \mathfrak{T}']$, and again we see a transition from 0 to negative at A'_*.

Note that A'_* is larger than A_*—i.e., the critical parameter value for the saddle-node-like bifurcation *changes* when we consider the system over $[0, \mathfrak{T}']$ vs when we consider the system over the whole time-interval $[0, \mathfrak{T}]$. This further shows how long-time-asymptotic approaches to considering stability (such as traditional Lyapunov exponents as opposed to finite-time Lyapunov exponents) are completely unsuitable for attempting to analyze the behavior of systems like the one we have just examined.

C. Our goal and main non-trivialities in achieving it

We seek to obtain a rigorous mathematical description, justification and generalization for the above-described stabilization phenomenon exhibited by Eqs. (7) and (9) and the quantification of stability given by Eq. (8).

In seeking to achieve this, the following three non-trivialities arise:

- As we have discussed in Sec. II B, the traditional long-time-asymptotic framework of stability analysis and stability quantification via Lyapunov exponents—as used, for example, to describe the stabilization phenomena in Secs. II A 1 and II A 2—cannot generally be used to describe the stabilization phenomenon in Eq. (7) or (9). So, we need to formulate new mathematical definitions of stability within a suitable finite-time-dynamics framework.
- The reasoning of Refs. 1 and 2 requires that while $G(t) \notin [-a, a]$, there is no significant net stabilizing or destabilizing of trajectories. It is clear that this is true over timescales comparable with the "fast" timescale of the time-frozen vector field: all trajectories move approximately periodically round the circle with common approximate period $\frac{2\pi}{\sqrt{G(t)^2-a^2}}$. However, it is somewhat less obvious that no significant net stabilizing or destabilizing of trajectories can eventually build up over the "slow" timescale of the forcing.
- If $G(t)$ passes in and out of the interval $(-a, a)$ multiple times, then each new time that $G(t)$ enters the interval $(-a, a)$, the previously synchronized solutions may experience some level of desynchronization due to a *canard* phenomenon[17] in which the synchronized solutions spend a long time following the slow motion of *moving source* $z_{G(t)} := \pi - y_{G(t)}$. This phenomenon requires extremely fine tuning of parameters but we shall see that it still makes it impossible for us to take a "true" limit as $\varepsilon \to 0$ in our slow–fast analysis. Rather, as we will prove, we will need that our "limit as $\varepsilon \to 0$" is allowed to pass over an exponentially vanishing set of "bad" ε-intervals.

The way in which we address these issues is essentially by adapting the methods used in Ref. 17 to investigate periodic orbits of (9) when $g(t)$ is sinusoidal. However, we will develop a much more elementary and more explicitly constructive approach to obtaining the necessary results about behavior while $G(t) \notin [-a, a]$ than that used in Ref. 17.

In addition to the above issues, our general results will require us to take into account another dynamical phenomenon that cannot occur in the Adler equation, namely, bifurcation-induced tipping[16,41,42] between stable equilibria of the time-frozen vector fields $\theta \mapsto F(\theta, \varepsilon t)$. This is not necessary when considering a forced Adler equation (7) because the time-frozen vector fields $\theta \mapsto -a\sin(\theta) + G(t)$ have at most one stable equilibrium. However, the presence of such tipping will not add much complication to the proofs.

III. SETUP, DEFINITIONS, AND RESULTS

Throughout this paper, we identify the circle $\mathbb{S}^1$ with $\mathbb{R}/(2\pi\mathbb{Z})$. (Hence, each tangent space $T_x\mathbb{S}^1$ is naturally identified with $\mathbb{R}$.)

An *arc* will always mean a connected proper subset of $\mathbb{S}^1$ with non-empty interior. For any two distinct points $a, b \in \mathbb{S}^1$, we define the "open arc extending anticlockwise from a to b"—written as (a, b)—to be the arc obtained by projecting an interval $(a', b') \subset \mathbb{R}$ onto $\mathbb{S}^1$, where $a' \in \mathbb{R}$ is any lift of a and b' is the unique lift of b in the interval $(a', a' + 2\pi)$. For such an open arc (a, b), we define $\mathrm{length}((a, b)) := b' - a'$; and for a non-open arc J, we set $\mathrm{length}(J) := \mathrm{length}(J^\circ)$. Just as $(a, b) \subset \mathbb{S}^1$ denotes an open arc, so likewise we use the analogous notations $[a, b)$, $(a, b]$, and $[a, b]$ for half-open arcs and closed arcs. For any $a \in \mathbb{S}^1$ and $x \in \mathbb{R}$, we write $a + x \in \mathbb{S}^1$ for the projection of $a' + x \in \mathbb{R}$, where $a' \in \mathbb{R}$ is any lift of a.

For any points $a, b \in \mathbb{S}^1$, we define the unsigned distance

$$d(a,b) := \begin{cases} \min(\mathrm{length}((a,b)), \mathrm{length}((b,a))), & a \neq b, \\ 0, & a = b. \end{cases}$$

For $p \in \mathbb{S}^1$ and $\delta > 0$, write $B_\delta(p) = \{\theta \in \mathbb{S}^1 : d(\theta, p) < \delta\}$. So, if $\delta \in (0, \pi)$, then $B_\delta(p) = (p - \delta, p + \delta)$.

Generally speaking, a variable θ will denote position on the circle $\mathbb{S}^1$, and a variable τ will denote "slow time," which in this paper belongs to the unit interval $[0, 1]$. Given a function $F\colon \mathbb{S}^1 \times [0, 1] \to \mathbb{R}$, we write $\frac{\partial^{m+n}F}{\partial\theta^m\partial\tau^n}$ $(m, n \geq 0)$ to denote the nth partial derivative with respect to the second input of the mth partial derivative with respect to the first input of F.

Most results in this section will be proved in Sec. V.

A. Notions of stability for slow–fast FTDSs on the circle

We denote by $\mathscr{F}$ the set of functions $F\colon \mathbb{S}^1 \times [0, 1] \to \mathbb{R}$ for which the partial derivatives $\frac{\partial F}{\partial\theta}$, $\frac{\partial F}{\partial\tau}$, $\frac{\partial^2 F}{\partial\theta^2}$, and $\frac{\partial^2 F}{\partial\theta\partial\tau}$ all exist and are continuous on the whole of $\mathbb{S}^1 \times [0, 1]$. We equip $\mathscr{F}$ with its natural norm,

$$\|F\|_{\mathscr{F}} = \max\left\{\|F\|_\infty, \left\|\tfrac{\partial F}{\partial\theta}\right\|_\infty, \left\|\tfrac{\partial F}{\partial\tau}\right\|_\infty, \left\|\tfrac{\partial^2 F}{\partial\theta^2}\right\|_\infty, \left\|\tfrac{\partial^2 F}{\partial\theta\partial\tau}\right\|_\infty\right\},$$

where $\|\cdot\|_\infty$ denotes the supremum norm for continuous functions on $\mathbb{S}^1 \times [0, 1]$.

Throughout this paper, we consider the differential equation

$$\dot\theta(t) = F(\theta(t), \varepsilon t), \quad t \in [0, \tfrac{1}{\varepsilon}], \tag{12}$$

where $F \in \mathscr{F}$.

Given an arc $J_0 \subset \mathbb{S}^1$, for any $\varepsilon > 0$ and $t \in \left[0, \frac{1}{\varepsilon}\right]$, we define the arc $J_t^{\varepsilon,F}$ to be the image of J_0 under the time-0-to-t mapping of (12), that is,

$$J_t^{\varepsilon,F} := \{\theta(t) \,:\, \theta(\cdot) \text{ is a solution of (12)}, \theta(0) \in J_0\}.$$

As in traditional dynamical systems theory, one can formulate *local* notions of stability around trajectories starting at individual initial conditions, as well as *global-scale* notions of stability that concern the whole ensemble of trajectories of all the initial conditions. Throughout this paper, we only consider such global-scale concepts of stability (and, therefore, we do not include the word "global" in our definitions).

Definition 1. We will say that F is *neutrally stable* if there exists $c \geq 1$ such that for every $\varepsilon > 0$ and $s, t \in \left[0, \frac{1}{\varepsilon}\right]$, for every arc J_0,

$$\frac{1}{c} \leq \frac{\mathrm{length}(J_t^{\varepsilon,F})}{\mathrm{length}(J_s^{\varepsilon,F})} \leq c. \tag{13}$$

Definition 2. We will say that F is *robustly neutrally stable* if there exists $c \geq 1$ and a neighborhood $\mathscr{U}$ of F in $\mathscr{F}$ such that for every $\tilde{F} \in \mathscr{U}$, for every $\varepsilon > 0$, $s, t \in [0, \frac{1}{\varepsilon}]$ and arc J_0,

$$\frac{1}{c} \leq \frac{\mathrm{length}(J_t^{\varepsilon,\tilde{F}})}{\mathrm{length}(J_s^{\varepsilon,\tilde{F}})} \leq c.$$

Definition 3. We will say that F is *exponentially stable with rate* $\Lambda \in (-\infty, 0)$ if given any $\eta, \Delta > 0$, for sufficiently small ε, there is an arc P_ε with $\mathrm{length}(P_\varepsilon) < \Delta$ such that for every arc J_0 not intersecting P_ε,

$$e^{\frac{\Lambda-\eta}{\varepsilon}} \leq \frac{\mathrm{length}(J_{1/\varepsilon}^{\varepsilon,F})}{\mathrm{length}(J_0)} \leq e^{\frac{\Lambda+\eta}{\varepsilon}}. \tag{14}$$

Due to canard phenomena, the exponential stability formulated in Definition 3 will often be too strong, and so we will need to exclude a small set of ε-values. In the following, we write $\mathrm{Leb}(\cdot)$ for the Lebesgue measure on $(0, \infty)$.

Definition 4. We will say that a set $\mathscr{E} \subset (0, \infty)$ is *exponentially vanishing* if there exists $\kappa \in (0, 1)$ such that for all sufficiently small $h > 0$,

$$\mathrm{Leb}((0, h] \cap \bar{\mathscr{E}}) \leq \kappa^{\frac{1}{h}}.$$

As an equivalent formulation, $\mathscr{E}$ is exponentially vanishing if and only if there exists $\kappa \in (0, 1)$ such that for all sufficiently large $\xi > 0$,

$$\mathrm{Leb}(\{\zeta \in [\xi, \infty) : \tfrac{1}{\zeta} \in \bar{\mathscr{E}}\}) \leq \kappa^{\xi}.$$

Definition 5. We will say that F is *almost exponentially stable with rate* $\Lambda \in (-\infty, 0)$ if given any $\eta, \Delta > 0$, there is an exponentially vanishing set $\mathscr{E} \subset (0, \infty)$ such that for every $\varepsilon \in (0, \infty) \setminus \mathscr{E}$, there is an arc P_ε with $\mathrm{length}(P_\varepsilon) < \Delta$ such that for every arc J_0 not intersecting P_ε, Eq. (14) holds.

All four stability definitions above can equivalently be formulated in terms of FTLEs. For an initial condition $\theta_0 \in \mathbb{S}^1$, letting $\theta(\cdot)$ be the solution of (12) starting at $\theta(0) = \theta_0$, we define the corresponding FTLE over a time-window $[s, t]$ by

$$\lambda_{s,t}^{\varepsilon,F}(\theta_0) := \frac{1}{t-s} \int_s^t \tfrac{\partial F}{\partial \theta}(\theta(u), \varepsilon u)\, du. \tag{15}$$

Note that $\int_s^t \frac{\partial F}{\partial \theta}(\theta(u), \varepsilon u)\, du$ is precisely the logarithm of the derivative at $\theta(s)$ of the time-s-to-t mapping of (12). Hence, by the mean value theorem, we immediately have the following:

Proposition 6. *For any $r > 0$, $\varepsilon > 0$, $s, t \in [0, \frac{1}{\varepsilon}]$ with $s < t$, and any connected subset S of $\mathbb{S}^1$ with non-empty interior, the following two statements are equivalent*:

- *for every arc $J_0 \subset S$,*

$$\frac{\mathrm{length}(J_t^{\varepsilon,F})}{\mathrm{length}(J_s^{\varepsilon,F})} \leq r;$$

- *for every $\theta_0 \in S$,*

$$\lambda_{s,t}^{\varepsilon,F}(\theta_0) \leq \tfrac{1}{t-s} \log r.$$

Likewise, the corresponding two statements with $\geq$ in place of $\leq$ are equivalent to each other.

Thus, in Definition 1, the statement that Eq. (13) holds for every arc J_0 can be reformulated as saying that for every $\theta_0 \in \mathbb{S}^1$,

$$|\lambda_{s,t}^{\varepsilon,F}(\theta_0)| \leq \tfrac{1}{t-s} \log c,$$

and similarly, in Definition 3 or Definition 5, the statement that Eq. (14) holds for every arc J_0 not intersecting P_ε can be reformulated as saying that for every $\theta_0 \in \mathbb{S}^1 \setminus P_\varepsilon$,

$$|\lambda_{0,\frac{1}{\varepsilon}}^{\varepsilon,F}(\theta_0) - \Lambda| \leq \eta.$$

B. Zeros, curves of the slow manifold, and fast connections

Throughout the rest of Sec. III, we assume that $F \in \mathscr{F}$.

Since we consider small ε, the dynamics of Eq. (12) will be essentially determined by the zeros of F.

Definition 7. A *zero of* F is a point $(\theta, \tau) \in \mathbb{S}^1 \times [0, 1]$ such that $F(\theta, \tau) = 0$. A zero (θ, τ) of F is called

- *hyperbolic stable* if $\frac{\partial F}{\partial \theta}(\theta, \tau) < 0$;
- *hyperbolic unstable* if $\frac{\partial F}{\partial \theta}(\theta, \tau) > 0$; and
- *non-hyperbolic* if $\frac{\partial F}{\partial \theta}(\theta, \tau) = 0$.

A non-hyperbolic zero (θ, τ) is called *non-degenerate* if $\frac{\partial^2 F}{\partial \theta^2}(\theta, \tau)$ and $\frac{\partial F}{\partial \tau}(\theta, \tau)$ are both non-zero.

Definition 8. A *curve of the stable* (respectively, *unstable*) *slow manifold* is a pair (U, y) consisting of a set $U \subset [0, 1]$ and a continuous function $y\colon U \to \mathbb{S}^1$ such that for all $\tau \in U$, $(y(\tau), \tau)$ is a hyperbolic stable (respectively, unstable) zero of F. We will also refer to the function y itself as being a *curve of the stable* (respectively, *unstable*) *slow manifold over* U.

We emphasize that in Definition 8, U is allowed to be the empty set (in which case y is the "empty function" $\emptyset \to \mathbb{S}^1$), or a disconnected set, and y need not admit a continuous extension to $\bar{U}$. In particular, if τ is a shared boundary point of two connected components of U and the limits $\lim_{\sigma \nearrow \tau} y(\sigma)$ and $\lim_{\sigma \searrow \tau} y(\sigma)$ both exist, these limits *do not* have to be equal to each other.

Remark 9. Let us now comment on the behavior of F locally around its zeros.

(A) Suppose (θ_0, τ_0) is a hyperbolic stable (respectively, unstable) zero of F. Then, by the implicit function theorem, there is a curve y of the stable (respectively, unstable) slow manifold over a neighborhood U of τ_0 such that every zero of F close to (θ_0, τ_0) lies on graph y.

(B) Suppose (θ_0, τ_0) is a non-degenerate non-hyperbolic zero of F with $\tau_0 \in (0, 1)$. Then, the family of vector fields $\{F(\cdot, \tau)\}_{\tau\in[0,1]}$ undergoes a *saddle-node bifurcation* near θ_0 as τ passes through τ_0 (see also the description at the start of Sec. III G). In particular, over an interval $U \subset [0, 1]$ with $\tau_0 \in \partial U$, there is both a curve y of the stable slow manifold and a curve z of the unstable slow manifold such that every zero of F close to (θ_0, τ_0) lies on the curve $\{(\theta_0, \tau_0)\} \cup \operatorname{graph} y \cup \operatorname{graph} z$.

Definition 10. Suppose we have $\tau \in [0, 1]$ and $\theta_1, \theta_2 \in \mathbb{S}^1$ such that (θ_1, τ) and (θ_2, τ) are zeros of F. We say that *there is a fast connection from* (θ_1, τ) to (θ_2, τ) if *at least one* of the following two statements holds:

- F is strictly positive on $(\theta_1, \theta_2) \times \{\tau\}$;
- F is strictly negative on $(\theta_2, \theta_1) \times \{\tau\}$.

C. Complete curves of the stable slow manifold

For a classical infinite-time autonomous differential equation $\dot{\theta} = f(\theta)$ with a stable equilibrium point y, one can consider the set of solutions that approach y arbitrarily closely as $t \to \infty$, and moreover, one can quantify the long-time-asymptotic stability of such solutions using the *Lyapunov exponent associated with y*, defined as $\frac{df}{d\theta}(y)$. [Likewise, for an infinite-time stationary-noise-driven system such as our example (6) of noise-induced stabilization, one can define analogous concepts,[43] with the analog of equilibrium points being invariant random Dirac measures.]

For our system (12), the analogous concept to a stable equilibrium point is what we will call a *complete curve of the stable slow manifold*.[44] Analogous to the arbitrarily close approach of an equilibrium point as $t \to \infty$ is the arbitrarily close *tracking* of such a curve as $\varepsilon \to 0$ (see Sec. III D); and in analogy to the Lyapunov exponent of an equilibrium, we will define the *adiabatic Lyapunov exponent* associated with a complete curve of the stable slow manifold, which (by Lemma 46) quantifies the stability of solutions that track the curve when ε is small.

First, we need to introduce a "genericity" assumption on F that will be required for us to be able to carry out the above-described analysis.

Definition 11. We will say that F is *generic* if

- every non-hyperbolic zero of F is non-degenerate;
- $\mathbb{S}^1 \times \{0, 1\}$ contains no non-hyperbolic zeros of F; and
- for each $\tau \in (0, 1)$, $\mathbb{S}^1 \times \{\tau\}$ contains at most one non-hyperbolic zero of F.

The first two points imply that every non-hyperbolic zero of F is as described in Remark 9(B).

If F is generic, then the number of zeros in $\mathbb{S}^1 \times \{0\}$ is even, with half being hyperbolic stable and half being hyperbolic unstable.

Definition 12. We will say that F is *initially multistable* if $\mathbb{S}^1 \times \{0\}$ contains at least two hyperbolic stable zeros.

Note, in particular, that if F has no zeros, then F is generic and not initially multistable.

The main results of this paper concern the stability properties of (12) when F is generic and not initially multistable. (If F *is* initially multistable, then F will generally not exhibit the "global-scale" stability properties formulated in Sec. III A.)

Definition 13. Suppose F is generic. A curve (U, y) of the stable slow manifold will be called *complete* if the following statements hold:

(a) U is open relative to $[0, 1]$ and has finitely many connected components;

(b) there are no zeros of F in $\mathbb{S}^1 \times ([0, 1] \setminus \bar{U})$; and

(c) for any $\tau \in (0, 1) \setminus U$ that is a shared boundary point of two connected components of U,
  - the limit $y(\tau-) := \lim_{\sigma\nearrow\tau} y(\sigma)$ exists and $(y(\tau-), \tau)$ is a non-hyperbolic zero of F;
  - the limit $y(\tau+) := \lim_{\sigma\searrow\tau} y(\sigma)$ exists and $(y(\tau+), \tau)$ is a hyperbolic stable zero of F; and
  - there is a fast connection from $(y(\tau-), \tau)$ to $(y(\tau+), \tau)$.

Theorem 14. *Suppose F is generic.*

(A) *If* $\mathbb{S}^1 \times \{0\}$ *contains no zeros of F, then there exists a unique complete curve* (U, y) *of the stable slow manifold. Furthermore, U is empty if and only if F has no zeros.*

(B) *For any* $\theta \in \mathbb{S}^1$ *with* $(\theta, 0)$ *being a hyperbolic stable zero of F, there exists a unique complete curve* (U_θ, y_θ) *of the stable slow manifold for which* $y_\theta(0) = \theta$.

We have the following immediate corollary:

Corollary 15. *If F is generic and not initially multistable, then there exists a unique complete curve of the stable slow manifold.*

The key method in the proof of Theorem 14 is *continuation* of a curve of zeros by virtue of Remark 9. The proof is given in Sec. V B.

Remark 16. As we will see from the proof of Theorem 14, if F is generic and initially multistable, then $\bar{U}_\theta$ is the same across all hyperbolic stable zeros $(\theta, 0)$ in $\mathbb{S}^1 \times \{0\}$; and letting C_1 be the connected component of $\bar{U}_\theta$ containing 0, we have that $U_\theta \setminus C_1$ and $y_\theta\big|_{U_\theta\setminus C_1}$ are the same across all hyperbolic stable zeros $(\theta, 0)$ in $\mathbb{S}^1 \times \{0\}$.

We now define adiabatic Lyapunov exponents, which generalize Eq. (8).

Definition 17. Suppose F is generic. Then, for each complete curve (U, y) of the stable slow manifold, we define the corresponding *adiabatic Lyapunov exponent* by

$$\Lambda_{\mathrm{ad}}(U, y) := \int_U \tfrac{\partial F}{\partial\theta}(y(\tau), \tau)\, d\tau. \tag{16}$$

In the case that F is not initially multistable, we simply write Λ_{ad} to refer to the adiabatic Lyapunov exponent associated with the unique complete curve of the stable slow manifold.

Note that $\Lambda_{\mathrm{ad}}(U, y)$ must be finite and non-positive and that $\Lambda_{\mathrm{ad}}(U, y) = 0$ if and only if $U = \emptyset$ (i.e., if and only if F has no zeros).

D. Tracking

So far, we have formulated various definitions connected with the zeros of F, but not yet connected with how this relates to the dynamics of (12). We now introduce notions of *tracking*, similar to that of Ref. 41.

In the following definitions, we assume that we have a set $V \subset [0,1]$ and a function $q\colon V \to \mathbb{S}^1$. Where we will apply these definitions, (V,q) will always be either a curve of the stable slow manifold or a curve of the unstable slow manifold.

Definition 18. Fix $\varepsilon > 0$. Given $\delta > 0$, we say that a function $\theta\colon \left[0, \frac{1}{\varepsilon}\right] \to \mathbb{S}^1$ *δ-tracks* (V,q) if for every $\tau \in V$, $d\left(\theta\left(\frac{\tau}{\varepsilon}\right), q(\tau)\right) < \delta$.

Definition 19. Fix $\varepsilon > 0$. Given $\Delta, \delta > 0$, we say that (12) *exhibits* (Δ,δ)*-tracking of* (V,q) if there is an arc P with $\mathrm{length}(P) < \Delta$ such that every solution $\theta(\cdot)$ of (12) with $\theta(0) \notin P$ δ-tracks (V,q).

Definition 20. We say that

- *F exhibits strict tracking of* (V,q) if given any $\Delta, \delta > 0$, we have that for sufficiently small ε, (12) exhibits (Δ,δ)-tracking of (V,q);
- *F almost exhibits strict tracking of* (V,q) if given any $\Delta, \delta > 0$, there is an exponentially vanishing set $\mathscr{E} \subset (0,\infty)$ such that for every $\varepsilon \in (0,\infty) \setminus \mathscr{E}$, (12) exhibits (Δ,δ)-tracking of (V,q); and
- *F potentially exhibits strict tracking of* (V,q) if given any Δ, $\delta > 0$, there is an open set $\mathscr{A} \subset (0,\infty)$ with $\inf \mathscr{A} = 0$ such that for every $\varepsilon \in \mathscr{A}$, (12) exhibits (Δ,δ)-tracking of (V,q).

Definition 21. We say that *F admits a global canard phenomenon* if there exists an open set $U \subset (0,1)$ and a curve z of the *un*stable slow manifold over U such that F potentially exhibits strict tracking of (U,z).

We now define what it means for F to exhibit "not necessarily strict" tracking of (V,q).

Definition 22. Assume that V is σ-compact. We say that *F exhibits* (respectively, *almost exhibits*, *potentially exhibits*) *tracking of* (V,q) if for every compact subset S of $V \setminus \{0\}$, F exhibits (respectively, almost exhibits, potentially exhibits) strict tracking of $(S, q|_S)$.

Remark 23. Suppose $V \subset (0,1]$, and we have functions $q_1, q_2\colon V \to \mathbb{S}^1$ that disagree on at least one point in the interior of V; then, we cannot have both that F exhibits tracking of q_1 and that F potentially exhibits tracking of q_2. However (as in Sec. III E), we may have that F almost exhibits tracking of q_1 and that F potentially exhibits tracking of q_2.

E. Main results

Standing assumption. Throughout this section, if F is generic and not initially multistable, then we take (U,y) to be the unique complete curve of the stable slow manifold, and we take Λ_{ad} to be the corresponding adiabatic Lyapunov exponent.

In our main result (Theorem 24), we consider functions F that are generic and not initially multistable and split into the following three cases: $\bar{U}$ has zero, one, and more than one connected component.

Theorem 24. (A) *Suppose F has no zeros. Then, F is robustly neutrally stable.*

(B) *Suppose F is generic and not initially multistable, and $\bar{U}$ is nonempty and connected. Then, F exhibits tracking of (U,y) and is exponentially stable with rate Λ_{ad}.*

(C) *Suppose F is generic and not initially multistable, and $\bar{U}$ is not connected. Then, F almost exhibits tracking of (U,y) and is almost exponentially stable with rate Λ_{ad}.*

Now, in case (C), a natural question is whether we have that F exhibits tracking of (U,y). In view of Remark 23, a negative answer is provided by the following result.

Proposition 25. *Suppose F is generic and not initially multistable and $\bar{U}$ is not connected. Then, F admits a global canard phenomenon.*

Specifically, letting τ_1 be the lower end point of the second connected component of $\bar{U}$, we will show that there exists $\tau' > \tau_1$ and a curve z of the unstable slow manifold over $(\tau_1,\tau']$ such that F potentially exhibits tracking of $((\tau_1,\tau'],z)$.

Now, let us consider case (A) further. Define

$$\mathfrak{c}_F(\varepsilon) := \sup_{\mathrm{arc}\, J_0} \max\left(\frac{\mathrm{length}(J^{\varepsilon,F}_{1/\varepsilon})}{\mathrm{length}(J_0)}, \frac{\mathrm{length}(J_0)}{\mathrm{length}(J^{\varepsilon,F}_{1/\varepsilon})}\right) \tag{17}$$

and

$$\mathfrak{r}_F(\varepsilon) := \log\, \mathfrak{c}_F(\varepsilon) = \max_{\theta_0 \in \mathbb{S}^1} \tfrac{1}{\varepsilon} |\lambda^{\varepsilon,F}_{0,\frac{1}{\varepsilon}}(\theta_0)|.$$

The neutral stability in case (A) obviously implies that $\mathfrak{c}_F(\varepsilon)$, and hence $\mathfrak{r}_F(\varepsilon)$, remains bounded as $\varepsilon \to 0$. This fact also follows immediately from Ref. 17, Theorem 2, but the proof of that result is highly involved and not explicitly constructive, meaning that one cannot easily write down a bound for these quantities as $\varepsilon \to 0$. However, our analysis of the behavior of (12) when F has no zeros will be more elementary and explicitly constructive, allowing us to obtain the following explicit bound.

Proposition 26. *Define the function $\mathfrak{e}_3\colon [0,\infty) \to \left[\frac{1}{6},\infty\right)$ by*

$$\mathfrak{e}_3(x) = \begin{cases} \frac{e^x - (1 + x + \frac{1}{2}x^2)}{x^3}, & x > 0, \\ \frac{1}{6}, & x = 0. \end{cases}$$

Suppose F has no zeros. For each $\tau \in [0,1]$, define

$$m_1(\tau) = \max_{\theta \in \mathbb{S}^1} \left|\tfrac{\partial F}{\partial \theta}(\theta,\tau)\right|,$$

$$m_2(\tau) = \max_{\theta \in \mathbb{S}^1} \left|\tfrac{\partial F}{\partial \tau}(\theta,\tau)\right|,$$

$$m_{11}(\tau) = \max_{\theta \in \mathbb{S}^1} \left|\tfrac{\partial^2 F}{\partial \theta^2}(\theta,\tau)\right|,$$

$$m_{12}(\tau) = \max_{\theta \in \mathbb{S}^1} \left|\tfrac{\partial^2 F}{\partial \theta \partial \tau}(\theta,\tau)\right|,$$

$$k(\tau) = \int_{\mathbb{S}^1} \frac{1}{|F(\theta,\tau)|}\, d\theta$$

and let

$$r(\tau) = m_{11}(\tau) m_2(\tau) k^2(\tau) \mathfrak{e}_3(m_1(\tau)k(\tau)) + \tfrac{1}{2} m_{12}(\tau) k(\tau).$$

Then,

$$\limsup_{\varepsilon \to 0} \mathfrak{r}_F(\varepsilon) \leq \int_0^1 r(\tau)\, d\tau + \min_{\tau \in [0,1]} m_1(\tau) k(\tau).$$

F. Sketches of the proofs and their relation to Ref. 17

Our results in Sec. III E are, in large part, obtained by adapting and generalizing methods used in Ref. 17 to study periodic orbits and their exhibition of the canard phenomenon in the periodic differential equation

$$\dot{\theta}(t) = -\cos(\theta) + k - \cos(\varepsilon t) \quad (1 < k < 2). \tag{18}$$

In particular, Theorem 24(B) of this paper is crudely analogous to Proposition 4(1) of Ref. 17, Theorem 24(C) of this paper is crudely analogous to Theorem 1(3) of Ref. 17, and Proposition 25 of this paper is crudely analogous to Theorem 1(4) of Ref. 17.

Let us now outline the basic structure of the set of proofs of results in Sec. III E.

First, foundational to the proofs are the basic formulas [Eqs. (33) and (34)] for the differentiable dependence of solutions on their initial condition and on ε, which we establish [along with our general notations for the solution mapping of (12)] in Sec. V A.

1. When F has no zeros

When F has no zeros, a kind of neutral stability is obtained in Ref. 17: namely, as we have already said, Theorem 2 of Ref. 17 immediately implies that $\mathfrak{c}_F(\varepsilon)$ [as in Eq. (17)] remains bounded as $\varepsilon \to 0$. However, it would be difficult or impossible to obtain from this proof an explicit bound on $\mathfrak{c}_F(\varepsilon)$, or to deduce from this proof that F is robustly neutrally stable in the sense of Definition 2.

So, in Sec. V C, we obtain Proposition 26 and Theorem 24(A) by a different approach. Specifically, our strategy is to approximate the nonautonomous differential equation (12) by a differential equation that is piecewise-autonomous on almost the whole time-interval, where on each "piece" all the solutions move strictly periodically round the circle with common period. The pieces are chosen to be of duration exactly equal to that common period, so that solutions do not mutually approach or move away from each other. Since $F(\cdot, \varepsilon t)$ has slow dependence on t, the piecewise-autonomous approximation is of high accuracy, and Grönwall's Lemma is used to obtain that even over the whole time-interval $\left[0, \frac{1}{\varepsilon}\right]$, the inaccuracy of the piecewise-autonomous approximation does not allow mutual synchronization of trajectories to build up.

Now, in addition to the neutral stability, a further issue that needs addressing is the dependence of $\theta(\frac{1}{\varepsilon})$ on ε given the initial condition $\theta(0)$, since this will ultimately play a central role in the proof of Theorem 24(C). In the proof of Lemma 2 of Ref. 17, this is addressed by building further upon the abovementioned Theorem 2 of Ref. 17. By contrast, in Corollary 45, we obtain the necessary result [namely, the existence of a strictly positive lower bound on the magnitude of the derivative of $\theta\left(\frac{1}{\varepsilon}\right)$ with respect to $\frac{1}{\varepsilon}$] as an immediate consequence of the fact that F is neutrally stable in the sense of Definition 1.

2. Tracking and stability

In cases (B) and (C) of Theorem 24, we have both a statement about tracking and a statement about stability. In fact, the statement about stability follows from the statement about tracking. This is fairly obvious when $\bar{U}$ is the whole interval $[0,1]$; and when $\bar{U}$ is a proper subset of $[0,1]$, we apply Theorem 24(A) to the subintervals of $[0,1]$ over which F has no zeros. Specifically, the statement linking tracking and stability is provided by Lemma 46 in Sec. V D.

3. Theorem 24(B)

In case (B) of Theorem 24, i.e., when $\bar{U}$ is connected, if $0 \notin U$, then it is not so obvious that F exhibits tracking of (U, y); specifically, the question arises of whether, as εt first enters into U, the trajectories of a significant proportion of initial conditions may exhibit the canard phenomenon of spending a long time tracking the curve of hyperbolic unstable zeros of F. So, we first study the case that $0 \in U$, in Sec. V E; and then [along somewhat similar lines to the proof of Proposition 4(1) of Ref. 17] for the case that $0 \notin U$, we apply the results of this to the time-reversal of F starting at a time slightly after the first entry into U and going back to time 0.

It is not very hard to show that F exhibits tracking of (U, y) when $\bar{U}$ is connected and $0 \in U$. In fact, letting $(z_0, 0)$ be the hyperbolic unstable zero of F in $\mathbb{S}^1 \times \{0\}$, we have (Proposition 50) that F exhibits tracking of (U, y) "away from z_0"; this is defined in Sec. V D and essentially means that the arc P in Definition 19 contains z_0 regardless of the value of ε. [By contrast, in the general definition of F exhibiting tracking of (U, y), the arc P is free to move around as $\varepsilon \to 0$.] As a result of this, by Lemma 46, we have (Proposition 51) that F is exponentially stable "away from z_0" with rate Λ_{ad}. More precisely, this notion is defined as follows:

Definition 27. Fix $p \in \mathbb{S}^1$. We say that F *is exponentially stable away from p with rate* $\Lambda \in (-\infty, 0)$ if given any $\eta, \Delta > 0$, for sufficiently small ε, for every arc J_0 not intersecting $B_{\frac{\Delta}{2}}(p)$, Eq. (14) holds.

Remark 28. By Proposition 6, Definition 27 is equivalent to saying that $\lambda^{\varepsilon,F}_{0,\frac{1}{\varepsilon}}(\cdot)$ converges to Λ uniformly on compact subsets of $\mathbb{S}^1 \setminus \{p\}$ as $\varepsilon \to 0$.

Section V F then deals with case (B) in the scenario that $0 \notin U$. To show that F exhibits tracking of (U, y), we apply Proposition 50 to deal with what happens after εt has entered U, and we apply the exponential stability result of Proposition 51 to the time-reversal of F as described above, in order to show that problems do not arise as εt enters into U. Again, Lemma 46 then gives that F is exponentially stable with rate Λ_{ad}.

4. Proposition 25

Having dealt with case (B), we next prove Proposition 25 in Sec. V G. We do this *before* proving Theorem 24(C), in order to illustrate the canard phenomenon that will need to be avoided outside an exponentially vanishing set of ε-values in Theorem 24(C). Reference 17 specifically looks for the situation that (18) has an almost-everywhere-attracting stable periodic orbit exhibiting the canard phenomenon; the fact that this occurs for intervals of ε-values arbitrarily close to 0 is proved by analyzing the time-$\frac{2\pi}{\varepsilon}$ mapping of (18). In a sense, our task of proving Proposition 25 is easier, in that we do not require any of the trajectories exhibiting the canard phenomenon to have any special additional property like "periodicity." [For example, the system (18) exhibits the canard phenomenon also for ε-values where there are no periodic orbits.] But, on the other hand, we are considering a free form of temporal dependence rather than a periodic dynamical system, and so the approach of simply analyzing the shape of the graph of a time-t mapping of the system cannot be applied.

The idea of the proof of Proposition 25 is as follows. Take a value σ that lies strictly in between the first and the second connected component of $\bar{U}$. The stability results that we have obtained so far for when $\bar{U}$ is connected can be applied over $[0,\sigma]$ (since $\bar{U}\cap[0,\sigma]$ is connected) and can also be applied in reverse time over $[\sigma,\tau']$. This gives us two high-density clusters of trajectories at time $\frac{\sigma}{\varepsilon}$, one of which moves unboundedly anticlockwise round the circle as ε decreases, while the other moves unboundedly clockwise round the circle as ε decreases (Proposition 52), such that the clusters cross each other infinitely many times as $\varepsilon\to 0$. [A similar idea is illustrated in Fig. 5(c) in Sec. IV E 2.] However, for the desired result, we need there to be arbitrarily small ε-values for which *not only* do these clusters meet each other, but actually the cluster obtained by evolving (12) forward in time over $\varepsilon t\in[0,\sigma]$ is entirely contained in the cluster obtained by evolving (12) backward in time over $\varepsilon t\in[\sigma,\tau']$. We achieve this by choosing τ' sufficiently close to the lower end point of the second connected component of $\bar{U}$ that the latter cluster shrinks at a slower rate than the former cluster as $\varepsilon\to 0$.

5. Theorem 24(C)

We prove Theorem 24(C) in Sec. V H by a similar approach to the proof in Ref. 17 that outside small intervals of ε-values system (18) has an almost-everywhere-attracting stable periodic orbit (Theorem 1, parts 1–3). In particular, Proposition 53 at the start of Sec. V H (which is an almost immediate consequence of Corollary 45 in Sec. V C) plays an analogous role to Lemma 2(1) of Ref. 17.

As before, the main task is to show that F almost exhibits tracking of (U,y), since Lemma 46 would then yield that F is almost exponentially stable with rate Λ_{ad}. Essentially, the idea is to show that in constructions like the one used to prove Proposition 25, the crossings between the clusters of trajectories only take place during an exponentially vanishing set of ε-values. Since the clusters are themselves shrinking at an exponential rate as $\varepsilon\to 0$, it is sufficient to find both a lower bound on the velocity (with respect to $\frac{1}{\varepsilon}$) at which the clusters move round the circle and an upper bound on the linear growth (again with respect to $\frac{1}{\varepsilon}$) of the cumulative distance of the circle swept out by the clusters (i.e., an upper bound on the "average velocity" of the clusters). The latter is fairly trivial [being essentially given by Eq. (35) in Sec. V A], while the former is essentially given by Proposition 53.

G. Transition between neutral stability and exponential stability

We now consider the typical behavior of a *transition* between case (A) and case (B) of Theorem 24 as a parameter of F is varied. First, let us recall the analogous scenario for autonomous dynamical systems.

Generically, for a (sufficiently smoothly) parameter-dependent autonomous dynamical system $\dot{\theta}=f_\gamma(\theta)$, a transition between the absence and presence of equilibrium points will take place via a "non-degenerate saddle-node bifurcation," where at the critical bifurcation parameter γ_0, letting θ_0 be the unique zero of f_{γ_0}, the following non-degeneracies hold:

- $d_1:=f'_{\gamma_0}(\theta_0)\neq 0$ and
- $d_2:=\frac{df_\gamma(\theta_0)}{d\gamma}\Big|_{\gamma=\gamma_0}\neq 0.$

The behavior of $f_\gamma(\theta)$ locally around $(\gamma=\gamma_0,\ \theta=\theta_0)$ is then similar to that of the *normal form*

$$f_\gamma^{\mathrm{normal}}(x)=\tfrac{1}{2}d_1x^2+d_2\gamma\quad(x\in\mathbb{R})$$

around $(\gamma=0,\ x=0)$. The zeros of this quadratic expression are precisely $x=\pm\sqrt{\frac{-2d_2\gamma}{d_1}}$ wherever this is well-defined, and the derivative of $f_\gamma^{\mathrm{normal}}$ at each such zero is given by $\pm\mathrm{sgn}(d_1)\sqrt{2d_1d_2\gamma}$. Accordingly, for the original parameter-dependent system $\dot{\theta}=f_\gamma(\theta)$, sufficiently close to the critical parameter value γ_0, we have the following:

(a) For γ on one side of γ_0, there are no equilibria and so all solutions move periodically round the circle with the same period.

(b) For γ on the other side of γ_0, there is one stable equilibrium y_γ and one unstable equilibrium z_γ; thus, almost all solutions converge to y_γ at an exponential rate with exponent $\lambda(\gamma):=f'_\gamma(y_\gamma)$. Furthermore,

$$\lambda(\gamma)\sim-\sqrt{2|d_1d_2(\gamma-\gamma_0)|}\tag{19}$$

as $\gamma\to\gamma_0$.

Now, for the slow–fast framework (12) with a parameter-dependent $F=F_\gamma$, the analogous scenario is a transition between the absence and presence of zeros of F_γ. [This is precisely what we see illustrated numerically in Figs. 2(a)–2(c) for the example

$$F_\gamma(\theta,\tau)=-a\sin(\theta)+k+\gamma g(\mathfrak{T}\tau),$$

and in Figs. 2(d)–2(f) for the example

$$F_\gamma(\theta,\tau)=-a\sin(\theta)+k+\gamma g(\mathfrak{T}'\tau),$$

with $g(\cdot)$ as in Sec. II B.] Assuming sufficient smoothness, generically a transition between the absence and presence of zeros of F_γ will be as follows:

At the critical parameter γ_0, F_{γ_0} has a unique zero (θ_0,τ_0), and $\tau_0\in(0,1)$ and the following non-degeneracies hold:

- the Hessian matrix
$$D_1=\begin{pmatrix}d_{\theta\theta}&d_{\theta\tau}\\d_{\theta\tau}&d_{\tau\tau}\end{pmatrix}:=\mathrm{Hess}_{F_{\gamma_0}}(\theta_0,\tau_0)$$
is invertible;
- $d_2:=\frac{dF_\gamma(\theta_0,\tau_0)}{d\gamma}\Big|_{\gamma=\gamma_0}\neq 0.$

As we will prove, the behavior locally around $(\gamma=\gamma_0,\ \theta=\theta_0,\ \tau=\tau_0)$ is then similar to that of the "normal form"

$$\begin{aligned}F_\gamma^{\mathrm{normal}}(\mathbf{x})&=\tfrac{1}{2}\mathbf{x}^\intercal D_1\mathbf{x}+d_2\gamma\quad(\mathbf{x}\in\mathbb{R}^2)\\&=\tfrac{1}{2}(d_{\theta\theta}x_1^2+d_{\tau\tau}x_2^2)+d_{\theta\tau}x_1x_2+d_2\gamma\end{aligned}\tag{20}$$

around $(\gamma=0,\ \mathbf{x}=(0,0))$. We will shortly consider the zeros of this quadratic expression and then state a corresponding result for

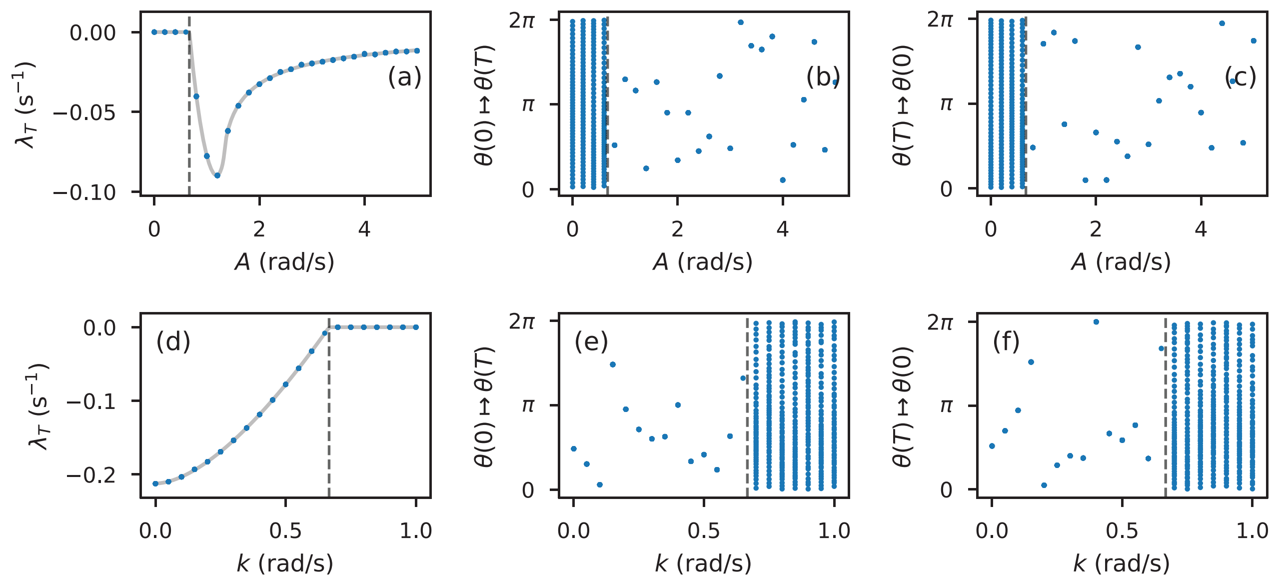

FIG. 3. Dynamics of (23), with varying A in (a)–(c) and varying k in (d)–(f). In all plots, $\omega = 10^{-3}$ rad/s, $T = 2\pi \times 10^5$ s, and $a = \frac{1}{3}$ rad/s. For (a)–(c), we have $k = 1$ rad/s, and the value $k - a = \frac{2}{3}$ rad/s is marked by the black dashed line. For (d)–(f), we have $A = \frac{1}{3}$ rad/s, and the value $a + A = \frac{2}{3}$ rad/s is marked by the black dashed line. In (a), (b), (d), and (e), results for the evolution $\theta(t)$ of 50 equally spaced initial conditions $\theta(0) = \frac{2\pi i}{50}$, $i = 0, \ldots, 49$, are shown: (a) and (d) show the finite-time Lyapunov exponents λ_T, as defined by (11), for these trajectories, and also show $\Lambda_{a,k,A}$ [defined in Eq. (31)] in gray; (b) and (e) show the positions $\theta(T)$ of these trajectories at time T. In (c) and (f), the positions of $\theta(0)$ for the 50 trajectories of (23) with $\theta(T) = \frac{2\pi i}{50}$, $i = 0, \ldots, 49$, are shown. Plots (a)–(c) reproduced with permission from Newman *et al.*, *Physics of Biological Oscillators*, edited by A. Stefanovska and P. V. E. McClintock (Springer, 2021), Chap. 7, pp. 111–129. Copyright 2021 Springer Nature Switzerland AG.

the original system (12), but first we can automatically say more about the matrix D_1.

Lemma 29. *Suppose we have $F \in C^2(\mathbb{S}^1 \times [0,1], \mathbb{R})$ with a zero $(\theta_0, \tau_0) \in \mathbb{S}^1 \times (0,1)$ such that $\nabla F(\theta_0, \tau_0) = (0,0)$ and $\det(\mathrm{Hess}_F(\theta_0, \tau_0)) < 0$. Then, F takes positive, negative, and zero values in an arbitrarily small punctured neighborhood of (θ_0, τ_0).*

Hence, in the above setup, we must have that $\det(D_1) > 0$. Note that for a 2×2 real symmetric matrix A with positive determinant, the diagonal entries are non-zero and have the same sign; so, we will write diag-sgn$(A) \in \{\pm 1\}$ for the sign of the diagonal entries of A.

Now, returning to consideration of the normal form (20), let us write

$$f_{\gamma,x_2}(x_1) = F_\gamma^{\mathrm{normal}}(x_1, x_2).$$

The discriminant of the quadratic function $f_{\gamma,x_2}(\cdot)$ is given by

$$\mathscr{D}(f_{\gamma,x_2}) = -\det(D_1)x_2^2 - 2\, d_{\theta\theta}\, d_2 \gamma.$$

We have that

- if $\mathscr{D}(f_{\gamma,x_2}) > 0$, then f_{γ,x_2} has two zeros, and the derivative of f_{γ,x_2} at these zeros is $\pm\sqrt{\mathscr{D}(f_{\gamma,x_2})}$;
- if $\mathscr{D}(f_{\gamma,x_2}) = 0$, then f_{γ,x_2} has one zero, and the derivative of f_{γ,x_2} at this zero is 0; and
- if $\mathscr{D}(f_{\gamma,x_2}) < 0$, then f_{γ,x_2} has no zeros.

Suppose without loss of generality that diag-sgn$(D_1) = 1$ and $d_2 > 0$. For $\gamma > 0$, $F_\gamma^{\mathrm{normal}}$ has no zeros. For $\gamma < 0$, we have

$$\tilde{U}_\gamma := \{x_2 : \mathscr{D}(f_{\gamma,x_2}) > 0\} = \left(-\sqrt{\frac{-2d_{\theta\theta}d_2\gamma}{\det(D_1)}}, \sqrt{\frac{-2d_{\theta\theta}d_2\gamma}{\det(D_1)}}\right) \neq \emptyset,$$

and so $F_\gamma^{\mathrm{normal}}$ has zeros as described above. The value of the integral $\int_{\tilde{U}_\gamma} -\sqrt{\mathscr{D}(f_{\gamma,x_2})}\, dx_2$ is the negative of the area of a semi-ellipse with principal radii $\sqrt{\frac{-2d_{\theta\theta}d_2\gamma}{\det(D_1)}}$ and $\sqrt{-2d_{\theta\theta}d_2\gamma}$, namely,

$$\int_{\tilde{U}_\gamma} -\sqrt{\mathscr{D}(f_{\gamma,x_2})}\, dx_2 = \frac{\pi\, d_{\theta\theta} d_2 \gamma}{\sqrt{\det(D_1)}}.$$

Now, let us make a rigorous statement about the original system (12) with dependence on a parameter γ.

We equip $C^2(\mathbb{S}^1 \times [0,1], \mathbb{R})$ with the standard C^2-topology. Fix an open interval $\Gamma \subset \mathbb{R}$, and let $\mathscr{F}_2$ be the set of all continuous paths $(F_\gamma)_{\gamma \in \Gamma}$ in the space $C^2(\mathbb{S}^1 \times [0,1], \mathbb{R})$ for which the functions

$$\frac{dF_\gamma}{d\gamma} : (\tilde{\gamma}, \theta, \tau) \mapsto \left.\frac{dF_\gamma(\theta,\tau)}{d\gamma}\right|_{\gamma=\tilde{\gamma}},$$

$$\frac{d(\nabla F_\gamma)}{d\gamma} : (\tilde{\gamma}, \theta, \tau) \mapsto \left.\frac{d(\nabla F_\gamma(\theta,\tau))}{d\gamma}\right|_{\gamma=\tilde{\gamma}}$$

are well-defined and continuous on $\Gamma \times \mathbb{S}^1 \times [0,1]$. (For each individual γ, we apply the same partial derivative notations to the function F_γ as were introduced at the start of Sec. III.)

Given $(F_\gamma)_{\gamma\in\Gamma} \in \mathscr{F}_2$, a point $(\gamma_0, \theta_0, \tau_0) \in \Gamma \times \mathbb{S}^1 \times [0, 1]$ is called a *critical zero* if $F_{\gamma_0}(\theta_0, \tau_0) = 0$ and $\nabla F_{\gamma_0}(\theta_0, \tau_0) = (0, 0)$. [This implies, in particular, that (θ_0, τ_0) is a degenerate non-hyperbolic zero of F_{γ_0}.] We say that a critical zero $(\gamma_0, \theta_0, \tau_0)$ is *non-degenerate* if $\mathrm{Hess}_{F_{\gamma_0}}(\theta_0, \tau_0)$ is invertible and $\frac{dF_\gamma}{d\gamma}(\gamma_0, \theta_0, \tau_0)$ is non-zero.

Theorem 30. *Suppose we have $(F_\gamma)_{\gamma\in\Gamma} \in \mathscr{F}_2$ with a non-degenerate critical zero $(\gamma_0, \theta_0, \tau_0)$ with $\tau_0 \in (0, 1)$, such that (θ_0, τ_0) is the only zero of F_{γ_0}. Let D_1 and d_2 be as above. Then, there exists $\delta > 0$ such that, defining the intervals Γ^- and Γ^+ by*

$$\Gamma^\bullet = \begin{cases} (\gamma_0 - \delta, \gamma_0) & \text{diag-sgn}(D_1) = \bullet\text{sgn}(d_2), \\ (\gamma_0, \gamma_0 + \delta) & \text{diag-sgn}(D_1) \neq \bullet\text{sgn}(d_2), \end{cases}$$

we have the following:

(a) *For all $\gamma \in \Gamma^-$, F_γ has no zeros and thus is neutrally stable.*
(b) *For all $\gamma \in \Gamma^+$, F_γ is generic with a unique complete curve (U_γ, y_γ) of the stable slow manifold, and U_γ is non-empty and connected. Thus, F_γ is exponentially stable with rate equal to the corresponding adiabatic Lyapunov exponent $\Lambda(\gamma)$. Furthermore,*

$$\Lambda(\gamma) \sim -\frac{\pi |d_{\theta\theta} d_2(\gamma - \gamma_0)|}{\sqrt{\det(D_1)}} \tag{21}$$

as $\gamma \to \gamma_0$.

Remark 31. Note that if $d_{\theta\tau} = 0$, then Eq. (21) simplifies to

$$\Lambda(\gamma) \sim -\pi \sqrt{\tfrac{d_{\theta\theta}}{d_{\tau\tau}}} |d_2(\gamma - \gamma_0)|. \tag{22}$$

Let us emphasize a key point of contrast between Eq. (21) and Eq. (19): namely, the latter has a square-root dependence on $\gamma - \gamma_0$ while the former has a linear dependence on $\gamma - \gamma_0$. So, although we said in Sec. II B that the bifurcation portraits in Fig. 2 bear resemblance of a classical saddle-node bifurcation, one significant way in which plots (a) and (d) differ from a generic saddle-node bifurcation is the following: as A rises above A_* and A'_*, respectively, the numerical FTLEs appear to grow in magnitude linearly (as opposed to the faster-than-linear growth that one would see in a generic saddle-node bifurcation).

Lemma 29 and Theorem 30 will be proved in Sec. V I.

IV. COMPARISON TO THE TRADITIONAL FRAMEWORK

In this section, we compare the stability formalisms and results in Sec. III to the traditional long-time-asymptotic framework:

- We will present definitions of neutral stability and exponential stability for systems on $\mathbb{S}^1$ that are closely analogous to Definitions 1 and 27, except now within the traditional $t \to \infty$ framework rather than our $\varepsilon \to 0$ framework.
- Whereas Secs. II A 1 and II A 2 considered stabilization of the Adler equation by constant forcing and by noise, and described the stabilization in terms of traditional $t \to \infty$ concepts, now we will consider stabilization of the Adler equation by *low-frequency periodic forcing* (especially sinusoidal[17,45–47] forcing) so that *both* the traditional $t \to \infty$ framework *and* (by restricting to finite time) the framework of Sec. III can be applied.

Remark 32. In any real-world situation, one will be working with a finite-duration process (as opposed to taking $t \to \infty$) subject to "finite-timescale" forcing (as opposed to taking $\varepsilon \to 0$). So, for a model in which both frameworks make sense, such as a low-frequency-periodically forced system, the question of which of the two frameworks (if either) provides a suitable approximation would need to be determined by context.

First, let us note that the discussion of Eq. (9) in Sec. II A 3 can be applied to the case that $g(t) = \cos(\omega t)$ for small ω: If $k > a$, and if we consider the system

$$\dot{\theta}(t) = -a \sin(\theta(t)) + k + A\cos(\omega t) \tag{23}$$

over a time-interval of duration greater than $\frac{\pi}{\omega}$, then the reasoning of Refs. 1 and 2 (as formalized by Theorem 24) gives that

- if $0 \leq A < k - a$, then system (23) exhibits neutrally stable dynamics (i.e., no significant mutual synchronization or repulsion of trajectories), and
- if $A > k - a$, then the Adler equation is stabilized: the trajectories of different initial conditions mutually synchronize into a tight cluster.

In Sec. IV A, we will define some general dynamical properties, especially stability properties, for infinite-time dynamical systems on the circle, within the classical long-time-asymptotic framework.

In Sec. IV B, we will apply these long-time-asymptotic concepts given in Sec. IV A to provide a basic stability analysis of periodically forced Adler equations. In particular, we will consider saddle-node bifurcations of periodic orbits and stabilization of the Adler equation with $k > a$.

In Sec. IV C, we will use numerical bifurcation diagrams of the same type as in Sec. II B to look at the parameter-dependence of the stability properties of low-frequency-sinusoidally forced Adler equations.

In Sec. IV D, we analyze Eq. (23) over finitely many periods of the forcing in terms of our new framework of Sec. III (in particular, applying Theorem 24). We then consider the numerics of Sec. IV C in the light of this theoretical analysis.

In Sec. IV E, we tie together all the above theoretical and numerical results to discuss how analysis of dynamics under the traditional framework and analysis of dynamics under our new framework relate and compare to each other for periodically forced Adler equations.

A. General notions of stability in the long-time-asymptotic framework

We consider systems of the form

$$\dot{\theta}(t) = F(\theta(t), t), \quad t \in [0, \infty) \tag{24}$$

for some continuous function $F\colon \mathbb{S}^1 \times [0, \infty) \to \mathbb{R}$ for which the partial derivatives $\frac{\partial F}{\partial \theta}$ and $\frac{\partial^2 F}{\partial \theta^2}$ exist and are continuous on the whole of $\mathbb{S}^1 \times [0, \infty)$. The most fundamental examples are the cases that

- (24) is *autonomous*, meaning that $F(\theta, t)$ is independent of t, i.e., $F(\theta, t) = f(\theta)$ for some $f\colon \mathbb{S}^1 \to \mathbb{R}$, and
- (24) is $\frac{2\pi}{\omega}$*-periodic* for some $\omega > 0$, meaning that $F(\theta, \cdot)$ is $\frac{2\pi}{\omega}$-periodic for all $\theta \in \mathbb{S}^1$.

More advanced examples include quasiperiodic systems, almost-periodic[48] systems, systems subject to forcing from a stationary stochastic process or measure-preserving dynamical system,[43] and asymptotically autonomous[41,49] systems. In the periodic case, we will write $\Phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ for the time-0-to-$\frac{2\pi}{\omega}$ mapping of (24), i.e., $\Phi\left(\theta\left(\frac{2\pi n}{\omega}\right)\right) = \theta\left(\frac{2\pi(n+1)}{\omega}\right)$ for any solution $\theta(\cdot)$ of (24) and any $n \in \mathbb{N}$.

Now, given an arc $J_0 \subset \mathbb{S}^1$, for any $t \in [0,\infty)$, we define the arc J_t^F to be the image of J_0 under the time-0-to-t mapping of (24), that is,

$$J_t^F := \{\theta(t) \,:\, \theta(\cdot) \text{ is a solution of (24)}, \theta(0) \in J_0\}.$$

For any $\theta_0 \in \mathbb{S}^1$ and $t > s \geq 0$, we define the finite-time Lyapunov exponent

$$\lambda_{s,t}^F(\theta_0) := \frac{1}{t-s}\int_s^t \tfrac{\partial F}{\partial \theta}(\theta(u),u)\,du, \tag{25}$$

where $\theta(\cdot)$ is the solution of (24) with $\theta(0) = \theta_0$. We define the (*asymptotic*) *Lyapunov exponent* by

$$\lambda_\infty^F(\theta_0) := \lim_{t\to\infty} \lambda_{s,t}^F(\theta_0), \tag{26}$$

if this limit exists in $\bar{\mathbb{R}}$; the existence and the value of the limit are clearly not dependent on s. We also define the (*asymptotic*) *average angular velocity* by

$$\Omega_F = \lim_{t\to\infty} \frac{\hat\theta(t)}{t},$$

where $\hat\theta\colon [0,\infty) \to \mathbb{R}$ may be any lift of any solution $\theta\colon [0,\infty) \to \mathbb{S}^1$ of (24), if this limit exists. Note that the existence and the value of this limit do not depend on the initial condition $\hat\theta(0)$: if $\hat\theta_1$ and $\hat\theta_2$ are lifts of solutions of (24), then $\sup_{t\geq 0}|\hat\theta_2(t) - \hat\theta_1(t)|$ cannot be larger than the smallest multiple of 2π above $|\hat\theta_2(0) - \hat\theta_1(0)|$. In the autonomous case $F(\theta,t) = f(\theta)$, we have

$$\Omega_F = \begin{cases} 0, & f \text{ has at least one zero,} \\ 2\pi\left(\int_{\mathbb{S}^1} \frac{1}{f(\theta)}\,d\theta\right)^{-1}, & f \text{ has no zeros.}\end{cases}$$

It is also well-known that in the periodic case, Ω_F exists and has continuous dependence (in the C^0-topology) on Φ.

Now, as in Sec. III, although one can formulate notions of stability and neutral stability locally around individual trajectories, here, we will just present global-scale notions of stability (and, therefore, we do not include the word "global" in our definitions).

Definition 33. We say that system (24) is *neutrally stable* if there exists $c \geq 1$ such that for every $s,t \in [0,\infty)$, for every arc J_0,

$$\frac{1}{c} \leq \frac{\text{length}(J_t^F)}{\text{length}(J_s^F)} \leq c. \tag{27}$$

Note that if system (24) is neutrally stable, then the Lyapunov exponent $\lambda_\infty^F(\theta_0)$ associated with every initial condition $\theta_0 \in \mathbb{S}^1$ is 0.

Remark 34. Suppose (24) is $\frac{2\pi}{\omega}$-periodic for some $\omega > 0$, and Ω_F is an irrational multiple of ω. Then, by Denjoy's theorem applied to Φ, (24) is neutrally stable.

Definition 35. Fix $p \in \mathbb{S}^1$. We say that the system (24) is *exponentially stable away from p with rate $\Lambda \in [-\infty,0)$* if given any $\eta, \Delta > 0$, for sufficiently large $t > 0$, for every arc J_0 not intersecting $B_{\frac{\Delta}{2}}(p)$, we have

$$\begin{cases} e^{(\Lambda-\eta)t} \leq \frac{\text{length}(J_t^F)}{\text{length}(J_0)} \leq e^{(\Lambda+\eta)t}, & \Lambda \neq -\infty, \\ \frac{\text{length}(J_t^F)}{\text{length}(J_0)} \leq e^{-\eta t}, & \Lambda = -\infty. \end{cases}$$

This is equivalent to saying that $\lambda_{0,t}^F(\cdot) \to \Lambda$ uniformly on compact subsets of $\mathbb{S}^1 \setminus \{p\}$ as $t \to \infty$. Hence, in particular, the Lyapunov exponent $\lambda_\infty^F(\theta_0)$ associated with each $\theta_0 \in \mathbb{S}^1 \setminus \{p\}$ is Λ.

Remark 36. Suppose (24) is $\frac{2\pi}{\omega}$-periodic for some $\omega > 0$ and that (24) is exponentially stable away from a point p with rate Λ. Then, considering the graph of Φ yields the following: the solution $\theta(\cdot)$ starting at $\theta(0) = p$ is $\frac{2\pi}{\omega}$-periodic; there is exactly one other $\frac{2\pi}{\omega}$-periodic solution; letting q be the initial condition of this latter periodic solution, we have $\Lambda = \lambda^F_{0,\frac{2\pi}{\omega}}(q)$; and Ω_F is an integer multiple of ω.

Typically, if (24) is $\frac{2\pi}{\omega}$-periodic and Ω_F is a non-integer rational multiple of ω, then (24) will admit multiple locally exponentially stable periodic solutions. (However, as we will soon see, this does not have to be the case.)

We have formulated the above definitions for differential equations but the same definitions also apply to infinite-time dynamical systems with non-differentiable solutions. For example, for system (6) presented in Sec. II A, there exists $p \in \mathbb{S}^1$ (dependent on W) such that (6) is exponentially stable away from p with some W-independent rate $\lambda_{a,k,A} < 0$.

B. A basic classical stability analysis of periodically forced Adler equations

We now apply the concepts of Sec. IV A to Adler equations with periodic additive forcing.

1. The possible behaviors

The following known result is essentially the main result of Ref. 45 (Theorems 1 and 4).

Proposition 37. *Take F in (24) of the form*

$$F(\theta,t) = -a\sin(\theta) + G(t),$$

where $a \in \mathbb{R}$ is constant and G is $\frac{2\pi}{\omega}$-periodic for some $\omega > 0$. Exactly one of the following three statements holds:

(i) *System (24) is neutrally stable.*
(ii) *For some $p \in \mathbb{S}^1$ and $\Lambda \in (-\infty,0)$, system (24) is exponentially stable away from p with rate Λ.*
(iii) *System (24) has exactly one $\frac{2\pi}{\omega}$-periodic solution $p(t)$; every solution $\theta(\cdot)$ of (24) has $d(\theta(t),p(t)) \to 0$ as $t \to \infty$; and $\lambda_\infty^F(\theta_0) = 0$ for all $\theta_0 \in \mathbb{S}^1$.*

Cases (ii) and (iii) can only occur if Ω_F is an integer multiple of ω; i.e., if Ω_F is not an integer multiple of ω, then system (24) is neutrally stable.

Case (iii) is a kind of "boundary case" between neutral stability and exponential stability; we expect that for "typical" a and $G(\cdot)$, if Ω_F is an integer multiple of ω then the system will be in case (ii).

2. Periodic saddle-node bifurcations

A transition between case (i) and case (ii) via case (iii) as some parameter is varied corresponds precisely to a saddle-node bifurcation of the time-0-to-$\frac{2\pi}{\omega}$ mapping Φ. Let us now describe one such scenario.

Proposition 38. *In Proposition 37, consider $G(t)$ of the form $G(t) = k + H(t)$, where we fix the function $H(\cdot)$ while considering varying values of $k \in \mathbb{R}$. Suppose we have $k_0 \in \mathbb{R}$ such that for $k = k_0$, system (24) is in case (iii) of Proposition 37. Then, sufficiently close to k_0, we have the following:*

- *for k on one side of k_0, system (24) is in case (i) of Proposition 37, i.e., is neutrally stable; and*
- *for k on the other side of k_0, system (24) is in case (ii) of Proposition 37, with*

$$\Lambda \sim -C\sqrt{|k-k_0|} \tag{28}$$

as $k \to k_0$ for some $C > 0$.

Note that the autonomous system (5) is precisely the case where $H \equiv 0$, and that the system (23) (for fixed A and ω) is precisely the case where $H(t) = A\cos(\omega t)$.

Recall that the autonomous case with $a > 0$ is as in Sec. II A 1: if $|k| > a$, then the system is neutrally stable, but if $|k| < a$, then the system is exponentially stable with exponent

$$\lambda_k := -\sqrt{a^2 - k^2} = -\sqrt{(a+k)(a-k)},$$

which approximates to $-\sqrt{2a|k-a|}$ for k near $\pm a$. This is precisely in accordance with the general description of saddle-node bifurcations of autonomous systems given at the start of Sec. III G. So, what Proposition 38 essentially says is that as we now go from the autonomous case to the periodic case, all k-parameterized transitions between neutral stability and exponential stability must still likewise take place via a "saddle-node bifurcation of $\frac{2\pi}{\omega}$-periodic orbits" exhibiting the same square-root dependence (28).

Proof of Proposition 38. Let $p(t)$ be as in Proposition 37 for $k = k_0$. By Ref. 45, Theorem 4, stereographic projection conjugates $\Phi_{(k=k_0)}$ to an orientation-preserving linear fractional transformation $\tilde{\Phi}$ of $\hat{\mathbb{R}}$ with a unique fixed point $\tilde{p}$; without loss of generality, we can take $\tilde{p} \in \mathbb{R}$, since otherwise we can first rotate the circle by angle π to get $\tilde{p} = 0$. The second derivative of any linear fractional transformation other than the identity function is non-zero throughout $\mathbb{R}$; and since smooth conjugation simply divides the second derivative at a fixed point by the derivative of the conjugacy at that point, it follows that $d_1 := \Phi''_{(k=k_0)}(p(0))$ is non-zero. Also, the derivative d_2 of the map $k \mapsto \Phi(p(0))$ at $k = k_0$ is given by $d_2 = \int_0^{\frac{2\pi}{\omega}} e^{\int_t^{\frac{2\pi}{\omega}} \frac{\partial F}{\partial \theta}(p(u),u)\,du}\,dt$, which is clearly strictly positive. Hence (similarly to the description at the start of Sec. III G), the map Φ undergoes a non-degenerate saddle-node bifurcation as k crosses k_0; and by Proposition 37, the two sides of the bifurcation correspond to cases (i) and (ii) of Proposition 37. Furthermore, in case (ii), we have that $\Lambda = \log D$, where D is the derivative of Φ at its unique stable fixed point, which [similarly to Eq. (19)] takes the form

$$D = 1 - \sqrt{2|d_1 d_2(k-k_0)|} + o(\sqrt{|k-k_0|}).$$

So,

$$\Lambda = -\sqrt{2|d_1 d_2(k-k_0)|} + o(\sqrt{|k-k_0|}).$$

□

3. Stabilization of the neutrally stable Adler equation

Let us now consider the question of *stabilization* of the Adler equation. Specifically, we consider the addition of zero-mean periodic forcing to the autonomous system $\dot{\theta} = -a\sin(\theta) + k$, where $k > a > 0$. Obviously, without the addition of any forcing [i.e., $G(t) = k$ in Proposition 37], the system belongs to case (i) of Proposition 37. The question is then whether, with the added zero-mean periodic forcing, the forced system now belongs to case (ii) of Proposition 37.

Let us first note that for the forced system, the average angular velocity must lie within the compact subinterval $[k-a, k+a]$ of $(0,\infty)$. This follows immediately from the fact that the term $-a\sin(\theta(t))$ always lies within the range $[-a,a]$. Since the average angular velocity has continuous dependence on the time-0-to-$\frac{2\pi}{\omega}$ mapping of the system, we immediately obtain from Proposition 37 the following corollary about low-frequency forcing: *there is no way to send ω to 0 without passing through the scenario that the system is neutrally stable.* Let us phrase this more precisely.

Corollary 39. *Fix $k > a > 0$. Consider the ε-parameterized periodic differential equation*

$$\dot{\theta}(t) = -a\sin(\theta(t)) + k + \tilde{g}(\varepsilon,t), \quad t \ge 0, \tag{29}$$

where $\tilde{g}\colon (0,\infty)\times\mathbb{R} \to \mathbb{R}$ is any continuous function with the following properties:

(a) *$\tilde{g}(\varepsilon,\cdot)$ is $\frac{2\pi}{\omega(\varepsilon)}$-periodic for some $\omega(\varepsilon) > 0$ with $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$, and*

(b) $\int_0^{\frac{2\pi}{\omega(\varepsilon)}} \tilde{g}(\varepsilon,t)\,dt = 0$.

System (29) is neutrally stable for an open set of ε-values arbitrarily close to 0.

Let us exemplify this with the prototypical case of sinusoidal forcing. Taking

$$\tilde{g}(\varepsilon,t) = A(\varepsilon)\cos(\omega(\varepsilon)t)$$

gives the following.

Corollary 40. *Fix $k > a > 0$. In the (A,ω)-parameter space of Eq. (23), it is not possible to plot a continuous path that tends toward the $\omega = 0$ axis without traveling through a region of the (A,ω)-parameter space where the system (23) is neutrally stable.*

In the most basic case, let us simply fix *any* value $A \in \mathbb{R}$: then, there is an open set of ω-values arbitrarily close to 0 for which system (23) is neutrally stable, i.e., for which *the forcing $+A\cos(\omega t)$ "fails to stabilize the Adler equation."*

C. Numerics for the sinusoidally forced Adler equation (23)

Having investigated periodically forced Adler equations such as (23) theoretically, we now plot numerical bifurcation diagrams for (23). We cannot "plot the long-time-asymptotic dynamics" simply through numerical simulation of trajectories since the simulations are themselves only of finite duration. However, as is typically done in numerical bifurcation diagrams, we can still plot "long-but-finite-time" results obtained from the finite-time simulations. (So, for example, we cannot calculate asymptotic Lyapunov exponents simply from numerically simulated trajectories, but we can calculate finite-time Lyapunov exponents taken over the entire duration of the simulation.)

Specifically, we will construct the same kind of plots as in Sec. II B. Fix $a = \frac{1}{3}$ rad/s (as in Sec. II B) and $\omega = 10^{-3}$ rad/s (which is the same as the cut-off angular frequency of the low-pass filter in Sec. II B). We will consider both

- varying A with fixed $k = 1$ rad/s $> a$ (as in Sec. II B) and
- varying k with fixed $A = \frac{1}{3}$ rad/s.

Note that ω is very small, i.e., we are considering low-frequency forcing; however, we still make sure that the duration of the numerical simulations is long enough to include many complete cycles of the low-frequency forcing. Specifically, we will work with 100 periods of the forcing [which is the same duration as the duration $\mathfrak{T}$ of the simulations represented in Figs. 2(a)–2(c) in Sec. II B]. In other words, we can think of the results as being for the 100th iterate of the circle map Φ. Results are shown in Fig. 3.

In Figs. 3(b) and 3(e), for the various combinations of A and k, the trajectories at time $T = \frac{200\pi}{\omega}$ of 50 evenly spaced initial conditions are shown. Figures 3(c) and 3(f) show the analogous results for the reverse-time system; due to the symmetries of (23), plots (c) and (f) turn out to be the reflection of plots (b) and (e) about $\frac{\pi}{2}$. Figures 3(a) and 3(d) show FTLEs for the trajectories shown in Figs. 3(b) and 3(e), respectively. These FTLEs are computed according to Eq. (11), which is precisely the same as Eq. (25) with $F(\theta, t) = -a\sin(\theta) + k + A\cos(\omega t)$.

Plots (a)–(c) show a picture somewhat resemblant of a saddle-node bifurcation: recalling that the system is neutrally stable for $A = 0$, we see the persistence of this exhibition of neutrally stable behavior for A going up to about $A = \frac{2}{3}$ rad/s, after which the system appears to be clearly stabilized.

Plots (d)–(f) likewise show a picture somewhat resemblant of a saddle-node bifurcation: we see stability for k going up to about $k = \frac{2}{3}$ rad/s, after which we see neutrally stable behavior. This is similar to the autonomous case $A = 0$, except with a higher critical k-threshold for the transition from stability to neutral stability.

However, plot (d) also reveals a feature distinctly different from a saddle-node bifurcation: as k approaches the critical threshold from below, the dependence of the finite-time Lyapunov exponents on k appears to be linear (with a gradient of about $\frac{1}{2}$) rather than of square-root form. [Plot (a) likewise appears potentially to indicate a linear dependence on A as the critical A-threshold is approached from above, but this is less clear.]

D. Stability analysis of finite-time low-frequency-sinusoidally forced Adler equations within the framework of Sec. III

Standing assumption. In all subsequent discussion of Eq. (23), we take $a > 0$ and $k, A \geq 0$.

We now consider Eq. (23) on finite time-intervals, within the slow–fast framework (12). Specifically, we will consider (23) over time-intervals $\left[0, \frac{n\pi}{\omega}\right]$, i.e., over $\frac{n}{2}$ cycles of the periodic forcing, for positive integers n. This can be expressed in the form of (12) by taking

$$F(\theta, \tau) = -a\sin(\theta) + k + A\cos(n\pi\tau), \tag{30}$$

with $\varepsilon = \frac{\omega}{n\pi}$. Obviously, F is not initially multistable. We have that F is generic if and only if $A \notin \{k + a, k - a, a - k\}$; in this case, we will denote the adiabatic Lyapunov exponent by $\Lambda_{a,k,A} \leq 0$, which clearly does not depend on n. We extend $\Lambda_{a,k,A}$ continuously to cover those (a, k, A)-values for which F is not generic. Explicitly,

$$\Lambda_{a,k,A} = \begin{cases} 0, & k \geq a \text{ and } A \leq k - a, \\ -\frac{1}{\pi}\int_0^\pi \sqrt{a^2 - (k + A\cos(t))^2}\, dt, & k < a \text{ and } A \leq a - k, \\ -\frac{1}{\pi}\int_{\arccos\left(\frac{a-k}{A}\right)}^{\pi} \sqrt{a^2 - (k + A\cos(t))^2}\, dt, & |a - k| < A \leq a + k, \\ -\frac{1}{\pi}\int_{\arccos\left(\frac{a-k}{A}\right)}^{\arccos\left(-\frac{a+k}{A}\right)} \sqrt{a^2 - (k + A\cos(t))^2}\, dt, & A > a + k. \end{cases} \tag{31}$$

Applying Theorem 24 gives the following.

Proposition 41. *For F as in Eq. (30):*

(A) *If $k > a$ and $A < k - a$, then F is neutrally stable.*
(B) *If*
- *$k < a$ and $A < a - k$, or*
- *$|a - k| < A < a + k$ and $n \in \{1, 2\}$, or*
- *$A > a + k$ and $n = 1$,*

then F is exponentially stable with rate $\Lambda_{a,k,A}$.
(C) *If*
- *$|a - k| < A < a + k$ and $n \geq 3$, or*
- *$A > a + k$ and $n \geq 2$,*

then F is almost exponentially stable with rate $\Lambda_{a,k,A}$.

Let us mention, in particular, the following cases:

Corollary 42. *Suppose $n \geq 3$.*

(A) *Fixing $k > a$ and letting $A_* := k - a$, we have*
- *if $A < A_*$ then F is neutrally stable, but*
- *if $A \in (A_*, \infty) \setminus \{k + a\}$ then F is almost exponentially stable with rate $\Lambda_{a,k,A}$.*

(B) *Fixing $a > 0$ and $A \geq 0$ and letting $k_* := a + A$, we have*

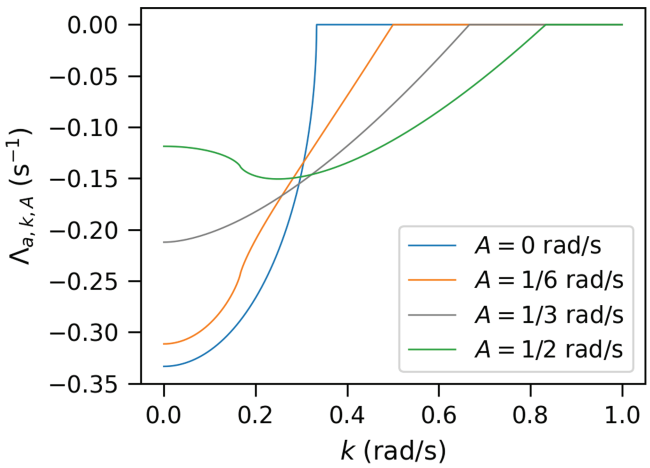

FIG. 4. Graph of $\Lambda_{a,k,A}$ [as defined in Eq. (31)] against k for different values of A, with $a = \frac{1}{3}$ rad/s. The blue curve $\Lambda_{a,k,0} = -\sqrt{\max(0, a^2 - k^2)}$ for $A = 0$ has infinite gradient on the left at $k = a$ where the curve transitions from negative to zero; the orange, gray, and green curves, respectively, have finite gradient $\frac{1}{\sqrt{2}}$, $\frac{1}{2}$, and $\frac{1}{2\sqrt{2}}$ on the left, where they transition from negative to zero (see Proposition 43).

- *if $k > k_*$, then F is neutrally stable, but*
- *if $k \in [0, k_*) \setminus \{|a - A|\}$, then F is almost exponentially stable with rate $\Lambda_{a,k,A}$ (with the "almost" not being necessary in the case that $A < a$ and $k < a - A$).*

We illustrate case (B) of Corollary 42 with a graph of $\Lambda_{a,k,A}$ against k for different values of A, in Fig. 4. One immediate observation from this figure is the following:

- for $A = 0$, i.e., the autonomous case (where a saddle-node bifurcation takes place as k crosses a), we see an infinite gradient in the graph as $k \nearrow a$, corresponding to the square-root dependence $\Lambda_{a,k,0} \approx -\sqrt{2a|k - a|}$ exactly as described just after the statement of Proposition 38; but
- when we now go to the nonautonomous periodic case $A > 0$, the graph has a finite gradient as $k \nearrow a + A$.

These finite gradients can be calculated using Theorem 30, as follows.

Proposition 43. (A) *Fixing $k > a$ and letting $A_* := k - a$, the map $A \mapsto \Lambda_{a,k,A}$ is right-sided-differentiable at A_* with derivative $-\frac{1}{2}\sqrt{\frac{a}{A_*}}$.*

(B) *Fixing $a, A > 0$ and letting $k_* := a + A$, the map $k \mapsto \Lambda_{a,k,A}$ is left-sided-differentiable at k_* with derivative $\frac{1}{2}\sqrt{\frac{a}{A}}$.*

Proof. In case (A), take F_γ to be the function F in Eq. (30) with $n = 2$ and $A = \gamma$. Then, taking $(\gamma_0, \theta_0, \tau_0) = \left(A_*, \frac{\pi}{4}, \frac{1}{2}\right)$, we have

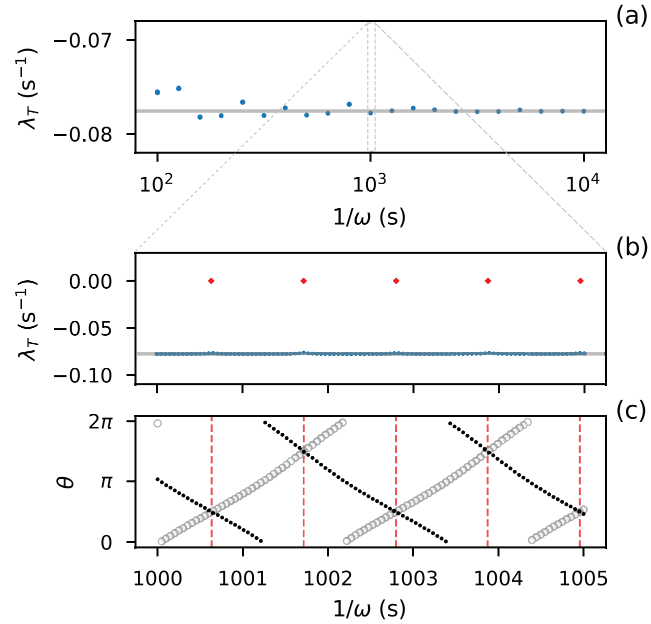

FIG. 5. Dynamics of (23) with varying ω. Other parameters are $a = \frac{1}{3}$ rad/s and $A = k = 1$ rad/s. (a) For each ω-value, taking $T = \frac{200\pi}{\omega}$ (i.e., 100 periods), the finite-time Lyapunov exponents λ_T as defined by (11) are shown for the trajectories of 50 equally spaced initial conditions $\theta(0) = \frac{2\pi i}{50}$, $i = 0, \ldots, 49$. The horizontal gray line marks the value of $\Lambda_{a,k,A}$ defined in (31). (b) Zoomed-in version of (a); the red points indicate the location of the small intervals of ω-values for which all initial conditions have zero asymptotic Lyapunov exponent. (c) Forward- and backward-time dynamics over the time-interval $\left[0, \frac{2\pi}{\omega}\right]$; for each ω-value, hollow circles show the positions of $\theta\left(\frac{2\pi}{\omega}\right)$ for the 50 trajectories of (23) with $\theta(0) = \frac{2\pi i}{50}$, $i = 0, \ldots, 49$, while solid circles show the positions of $\theta(0)$ for the 50 trajectories of (23) with $\theta\left(\frac{2\pi}{\omega}\right) = \frac{2\pi i}{50}$, $i = 0, \ldots, 49$. In both cases, the 50 points are clustered together; the values of ω where the two curves of clustered points cross correspond to where the red points are marked in (b); see Appendix B for further explanation.

that (θ_0, τ_0) is the only zero of F_{γ_0}, and

$$
\begin{aligned}
d_{\theta\theta} &= a, \\
d_{\tau\tau} &= 4\pi^2 A_*, \\
d_{\theta\tau} &= 0, \\
d_2 &= -1.
\end{aligned}
$$

Thus, the conditions of Theorem 30 are satisfied, and applying Eq. (22) gives the result. Case (B) is proved the same way, with $k = \gamma$ and $\gamma_0 = k_*$. □

Now, just as one can seek numerical help in investigating the long-time-asymptotic behavior of an infinite-time dynamical system through long-but-finite-time simulations, so likewise one can seek numerical help in investigating the $\varepsilon \to 0$ behavior of a dynamical system of form (2) through small-but-positive-ε simulations. Now, combining the two: suppose we have a finite-time numerical simulation of an infinite-time system subject to external forcing, where the timescale of the forcing in the simulation is much slower than the timescale of the system's internal dynamics, *and* the duration of the simulation is very large compared to the timescale of the forcing. Then, exactly the same comment that we made in Remark 32 regarding real-world processes also applies to the simulation: one can interpret it as either an approximation of the $t \to \infty$ behavior or an approximation of the $\varepsilon \to 0$ behavior. Note that Fig. 3 comes under exactly this category: ω is very small, but at the same time, n is very large.

Accordingly, let us now reconsider Fig. 3 in the light of Corollary 42. In all six plots, we have indicated the value of $A_* = k - a$ or $k_* = a + A$ as appropriate by a dashed black line. Additionally, in plots (a) and (d), we have shown in gray the value of $\Lambda_{a,k,A}$ as given in Eq. (31); so, the gray curve in plot (d) is the same as the gray curve in Fig. 4. The numerical results in Fig. 3 show perfect agreement with the theoretical predictions of Corollary 42 regarding stability, neutral stability, and quantification of stability, just as the numerics in Sec. II B for "unextendible" finite-time forcing likewise showed perfect agreement with the theoretical predictions of Theorem 24 regarding stability and neutral stability. (Recall that all these theoretical predictions are simply the mathematically precise description of what one would heuristically expect to see by applying the reasoning of Refs. 1 and 2.)

E. Commentary on the above results

So far, we have presented theoretical results within the long-time-asymptotic framework, numerical results, and theoretical results within the finite-time slow–fast framework. Let us now tie these together and draw conclusions.

1. Stable vs neutrally stable dynamics for $A > k - a$

At the end of Sec. IV B 3, we saw that when $k > a$, for *any* fixed value of A there are robust ω-values arbitrarily close to 0 for which the system (23) is neutrally stable in the sense of Sec. IV A. (Recall that this implies, in particular, that all trajectories have an asymptotic Lyapunov exponent of zero.) The proof of this result made no recognition whatsoever of a distinction between $A < k - a$ and $A > k - a$. Yet, Figs. 3(a)–3(c) show a very clear distinction between neutrally stable behavior for $A < k - a$ and stable behavior $A > k - a$ exactly as in Corollary 42. The fact that these ω-values of neutral stability exist arbitrarily close to zero even when $A > k - a$ indicates that, at such ω-values, the heuristic reasoning described in Sec. II A 3 must break down.

The cause of this breakdown is precisely the canard phenomenon where solutions spend a long time tracking the motion of the slowly moving source and thus experience mutual desynchronization. It is well-known that the canard phenomenon is an extremely fine-tuned phenomenon; in other words, the open set of ω-values described at the end of Sec. IV B 3 must consist of extremely narrow ω-intervals if $A > k - a$. Indeed, for the range of parameter-values considered in Ref. 17, it was proved that this set of ω-values is exponentially vanishing; and similarly in this present paper, we have shown [Theorem 24(B)–(C)] that for any fixed finite-length shape of slow-timescale variation, the set of timescale-separation values for which the canard phenomenon causes the reasoning in Sec. II A 3 to break down is likewise exponentially vanishing. However, the fact that the $A > k - a$ case requires this fine-tuned canard mechanism of desynchronization in order for the neutral stability (in the sense of Sec. IV A) to be achieved is altogether missed by the bifurcation and stability analysis given in Sec. IV B.

2. Illustration through further numerics

Let us now consider through further numerics how the dynamics of Eq. (23) depends on ω when $A > k - a$. Figures 5(a)–5(b) are FTLE plots similar to those in plots (a) and (d) of Figs. 2 and 3, except that the varied parameter is now ω rather than A or k. We take $a = \frac{1}{3}$ rad/s and $A = k = 1$ rad/s, so $k - a < A < k + a$, implying that F is exponentially stable for $n \in \{1, 2\}$ and almost exponentially stable for $n \geq 3$; here, as in Fig. 3, we take 100 cycles of the forcing, i.e., the results are shown over a duration of $\frac{200\pi}{\omega}$. In gray is shown the value of $\Lambda_{a,k,A}$. For ω sufficiently small, we see a close match of the FTLEs to $\Lambda_{a,k,A}$. In plot (b) are also indicated the approximate locations of the narrow ω-intervals for which system (23) is neutrally stable in the sense of Sec. IV A. (The procedure by which this is achieved is described in Appendix B.) Despite the high density of ω-values for which FTLEs are plotted in plot (b), all the results are close to $\Lambda_{a,k,A}$ and none are close to 0.

For further illustration, we now zoom in around one of the points marked in Fig. 5(b) as being the approximate location of an ω-interval of neutral stability. In Fig. 6, FTLE results are shown for both 100 cycles and 1000 cycles of the periodic forcing. Despite the extremely high density of ω-values for which results are shown, all the shown FTLEs approximate the value $\Lambda_{a,k,A} < 0$ to an accuracy of at least about 97%.

3. Dependence of Lyapunov exponents on k

The apparently linear dependence of the FTLEs in Fig. 3(d) to the left of $k = \frac{2}{3}$ rad/s is perfectly explained by Corollary 42 and Proposition 43: the left-sided derivative of $k \mapsto \Lambda_{a,k,A}$ at $k = a + A$ is equal to $\frac{1}{2}$. This is in contrast to Proposition 38, which gives us the following:

- for any transition between stability and neutral stability (understood within the framework of Sec. IV A) that arises from keeping the parameters a, A, and ω fixed while varying k past a critical value k_0, the Lyapunov exponent on the stable side of the transition must have *square-root* dependence on $|k - k_0|$ as k approaches k_0.

One question that then naturally arises is the following: if we fix ω, A, and a as in Figs. 3(d)–3(f), then over an interval of k-values such as, say, $[0.65, 1]$ rad/s, which k-values correspond to neutral stability and which k-values correspond to exponential stability with small Lyapunov exponent? Nonetheless, the fundamental mechanism of stabilization (as described in Sec. II A 3 and formalized by Theorem 24) that gives rise to what we see in Fig. 3 is essentially blind to the distinction between these two scenarios; indeed, it is exactly this same mechanism of stabilization that also gave rise to the equally clear pictures seen in Fig. 2, where questions of long-time-asymptotic dynamics do not make sense.

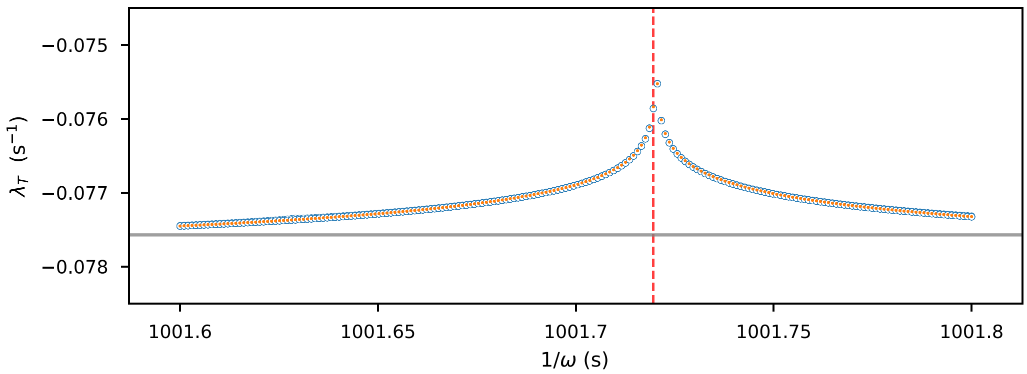

FIG. 6. Dynamics of (23) with varying ω near a red-marked point in Fig. 5(b). Parameters a, k, and A are as in Fig. 5. For each ω-value, in hollow blue are shown the finite-time Lyapunov exponents λ_T associated with the trajectories of 10 initial conditions $\theta(0) = \frac{2\pi i}{10}$ with $T = \frac{200\pi}{\omega}$ (i.e., 100 periods), and in solid orange are shown the finite-time Lyapunov exponents λ_T associated with the trajectories of three initial conditions $\theta(0) = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$ with $T = \frac{2000\pi}{\omega}$ (i.e., 1000 periods). The red dashed line indicates the approximate location of a small interval of ω-values for which all trajectories have zero asymptotic Lyapunov exponent. The horizontal gray line marks the value of $\Lambda_{a,k,A}$ defined in (31).

V. PROOFS OF RESULTS IN SEC. III

We begin with some further notations.

We will always assume that $F \in \mathscr{F}$. Define

$$M = \|F\|_\infty,\ M_1 = \left\|\tfrac{\partial F}{\partial \theta}\right\|_\infty,\ M_2 = \left\|\tfrac{\partial F}{\partial \tau}\right\|_\infty,$$

$$M_{11} = \left\|\tfrac{\partial^2 F}{\partial \theta^2}\right\|_\infty,\ M_{12} = \left\|\tfrac{\partial^2 F}{\partial \theta \partial \tau}\right\|_\infty.$$

Given $\sigma_1, \sigma_2 \in [0,1]$, define $F_{\sigma_1,\sigma_2} \in \mathscr{F}$ by

$$F_{\sigma_1,\sigma_2}(\theta,\tau) = (\sigma_2 - \sigma_1)F(\theta, \sigma_1 + (\sigma_2 - \sigma_1)\tau).$$

So, for fixed ε, if $\theta(\cdot)$ is a solution of Eq. (12), then

$$t \mapsto \theta\left(\tfrac{\sigma_1}{\varepsilon} + (\sigma_2 - \sigma_1)t\right)$$

is a solution of Eq. (12) with F_{σ_1,σ_2} in place of F: denoting this function by $\theta_{\sigma_1,\sigma_2}$, we have

$$\begin{aligned}\dot\theta_{\sigma_1,\sigma_2}(t) &= (\sigma_2 - \sigma_1)\dot\theta\left(\tfrac{\sigma_1}{\varepsilon} + (\sigma_2 - \sigma_1)t\right)\\ &= (\sigma_2 - \sigma_1)F\left(\theta_{\sigma_1,\sigma_2}(t), \varepsilon\left(\tfrac{\sigma_1}{\varepsilon} + (\sigma_2 - \sigma_1)t\right)\right)\\ &= (\sigma_2 - \sigma_1)F(\theta_{\sigma_1,\sigma_2}(t), \sigma_1 + (\sigma_2 - \sigma_1)\varepsilon t)\\ &= F_{\sigma_1,\sigma_2}(\theta_{\sigma_1,\sigma_2}(t), \varepsilon t).\end{aligned}$$

Note that if F is generic and $\sigma_1 \neq \sigma_2$, then F_{σ_1,σ_2} is also generic.

Recall the finite-time Lyapunov exponents $\lambda^{\varepsilon,F}_{s,t}(\theta_0)$ defined in Eq. (15); for convenience, also define the notation

$$\begin{aligned}\bar\lambda^{\varepsilon,F}_{s,t}(\theta_0) &:= \int_s^t \tfrac{\partial F}{\partial \theta}(\theta(u), \varepsilon u)\, du\\ &= (t-s)\lambda^{\varepsilon,F}_{s,t}(\theta_0),\end{aligned}$$

where the first equality is used to define $\bar\lambda^{\varepsilon,F}_{s,t}(\theta_0)$ even for $s \geq t$. Note that

$$\bar\lambda^{\varepsilon,F}_{s,t}(\theta_0) = \bar\lambda^{\varepsilon,F_{\varepsilon s,\varepsilon t}}_{0,\frac{1}{\varepsilon}}(\theta_0).$$

A. Solution mapping of (12) and its partial derivatives

Define the function

$$\Phi\colon (0,\infty) \times \mathbb{S}^1 \times [0,1] \times [0,1] \to \mathbb{S}^1,$$

such that, given any $\varepsilon > 0$, for any solution $\theta(\cdot)$ of (12) and any $s, t \in \left[0, \frac{1}{\varepsilon}\right]$, we have

$$\Phi\left(\tfrac{1}{\varepsilon}, \theta(s), \varepsilon s, \varepsilon t\right) = \theta(t).$$

Equivalently, given any $\varepsilon > 0$, for any solution $\phi\colon [0,1] \to \mathbb{S}^1$ of the differential equation

$$\dot\phi(t) = \tfrac{1}{\varepsilon}F(\phi(t), t) \tag{32}$$

and any $s, t \in [0,1]$,

$$\Phi\left(\tfrac{1}{\varepsilon}, \phi(s), s, t\right) = \phi(t).$$

We denote the partial derivative of Φ with respect to its first input and its second input, respectively, by

$$\tfrac{\partial \Phi}{\partial(\varepsilon^{-1})}, \tfrac{\partial \Phi}{\partial \theta}\colon (0,\infty) \times \mathbb{S}^1 \times [0,1] \times [0,1] \to \mathbb{R}.$$

We now give formulas for these partial derivatives.

Proposition 44. *Take any $\varepsilon > 0$ and let $\theta(\cdot)$ be a solution of (12) starting at $\theta(0) = \theta_0$. For all $s, t \in \left[0, \frac{1}{\varepsilon}\right]$,*

$$\tfrac{\partial \Phi}{\partial \theta}\left(\tfrac{1}{\varepsilon}, \theta(s), \varepsilon s, \varepsilon t\right) = e^{\bar\lambda^{\varepsilon,F}_{s,t}(\theta_0)}, \tag{33}$$

$$\tfrac{\partial \Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon}, \theta(s), \varepsilon s, \varepsilon t\right) = \varepsilon \int_s^t e^{\bar\lambda^{\varepsilon,F}_{u,t}(\theta_0)} F(\theta(u), \varepsilon u)\, du. \tag{34}$$

Proof. Equation (33) is a standard result about the differentiable dependence of solutions on their initial condition. Now, let $\phi(\tau) = \theta\left(\frac{\tau}{\varepsilon}\right)$ for each $\tau \in [0,1]$; so, ϕ satisfies Eq. (32). The derivative of the map $r \mapsto rF(\phi(\tau), \tau)$ at $r = \frac{1}{\varepsilon}$ is obviously just $F(\phi(\tau), \tau)$, and so we have

$$\begin{aligned}
&\tfrac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon},\phi(\varepsilon s),\varepsilon s,\varepsilon t\right)\\
&=\int_{\varepsilon s}^{\varepsilon t}\tfrac{\partial\Phi}{\partial\theta}\left(\tfrac{1}{\varepsilon},\phi(\tau),\tau,\varepsilon t\right)F(\phi(\tau),\tau)\,d\tau\\
&=\varepsilon\int_{s}^{t}\tfrac{\partial\Phi}{\partial\theta}\left(\tfrac{1}{\varepsilon},\theta(u),\varepsilon u,\varepsilon t\right)F(\theta(u),\varepsilon u)\,du\\
&=\varepsilon\int_{s}^{t}e^{\bar{\lambda}^{\varepsilon,F}_{u,t}(\theta_0)}F(\theta(u),\varepsilon u)\,du.
\end{aligned}$$

□

It will also be useful to define the function

$$\Delta\Phi\colon(0,\infty)\times\mathbb{S}^1\to\mathbb{R},$$

such that, given any ε, for any lift $\hat{\theta}\colon\left[0,\frac{1}{\varepsilon}\right]\to\mathbb{R}$ of a solution $\theta\colon\left[0,\frac{1}{\varepsilon}\right]\to\mathbb{S}^1$ of (12), we have

$$\Delta\Phi\left(\tfrac{1}{\varepsilon},\theta(0)\right)=\hat{\theta}\left(\tfrac{1}{\varepsilon}\right)-\hat{\theta}(0).$$

Since solutions $\theta(\cdot)$ of (12) have $|\dot{\theta}(t)|\le M$, we have that

$$|\Delta\Phi\left(\tfrac{1}{\varepsilon},\theta_0\right)|\le\tfrac{1}{\varepsilon}M \tag{35}$$

for all $\varepsilon>0$ and $\theta_0\in\mathbb{S}^1$.

B. Proof of Theorem 14

Assume that F is generic.

If F has no zeros, then it is clear that $(\emptyset,\emptyset\to\mathbb{S}^1)$ is a complete curve of the stable slow manifold (and obviously there is no other).

Now, suppose that F has zeros, and let $C=\{\tau\in[0,1]:\exists\,\theta\in\mathbb{S}^1\text{ s.t. }F(\theta,\tau)=0\}$, i.e., C is the image of the set of zeros of F under $(\theta,\tau)\mapsto\tau$. Since F is continuous, we obviously have that C is closed. Note that if we have a complete curve (U,y) of the stable slow manifold, then $C=\bar{U}$.

By Remark 9(A), if we have $\tau\in[0,1]$ such that $\mathbb{S}^1\times\{\tau\}$ includes a hyperbolic stable or unstable zero of F, then τ is in the interior of C relative to $[0,1]$. Hence, for every boundary point τ of C relative to $[0,1]$, $\mathbb{S}^1\times\{\tau\}$ only contains a non-hyperbolic zero of F.

Now, if there are infinitely many $\tau\in[0,1]$ for which $\mathbb{S}^1\times\{\tau\}$ contains a non-hyperbolic zero, then, in particular, we can find a convergent sequence of distinct non-hyperbolic zeros (θ_n,τ_n) converging to a zero (θ,τ); but by Remark 9, no zero of F can have non-hyperbolic zeros of F arbitrarily close to it. Hence, there are only finitely many $\tau\in[0,1]$ for which $\mathbb{S}^1\times\{\tau\}$ contains a non-hyperbolic zero. Hence, in particular, C has only finitely many boundary points, and, therefore, only finitely many connected components. Furthermore, by Remark 9, no connected component of C can be a singleton.

So, let $C_1,\dots,C_n$ be the connected components of C arranged in increasing order. For each connected component $C_i=[a_i,b_i]$ of C,

- if $a_i\neq 0$, then since $\mathbb{S}^1\times\{a_i\}$ contains exactly one zero of F and this zero is non-hyperbolic, on the basis of Remark 9(B) there must exist $\delta>0$ and a curve y of the stable slow manifold over $(a_i,a_i+\delta)$ such that for each $\tau\in(a_i,a_i+\delta)$, (y_τ,τ) is the only hyperbolic stable zero of F in $\mathbb{S}^1\times\{\tau\}$; but
- if $a_i=0$ (obviously implying, in particular, that $i=1$), then for each hyperbolic stable zero $(\theta_0,0)$ of F in $\mathbb{S}^1\times\{0\}$, on the basis of Remark 9(A), we have that for some $\delta>0$, there is a unique curve y of the stable slow manifold over $[0,\delta)$ for which $y(0)=\theta_0$.

In either case, let $\tau_i^{(1)}$ be the supremum of the set of all $\tau>a_i$ for which there is a unique curve of the stable slow manifold over (a_i,τ) agreeing with y on $(a_i,a_i+\delta)$; note that if $a_i=0$, then $\tau_i^{(1)}$ will depend on θ_0. It is clear that this "supremum" is indeed a maximum, i.e., that there is a unique curve of the stable slow manifold over $(a_i,\tau_i^{(1)})$ agreeing with y on $(a_i,a_i+\delta)$. So, let us extend the domain of y to include $(a_i,\tau_i^{(1)})$. Now, let σ_n be a strictly increasing sequence converging to this maximum $\tau_i^{(1)}$ such that $y(\sigma_n)$ is convergent to some limit l. We have that $(l,\tau_i^{(1)})$ is a zero of F; and so by Remark 9 together with the fact that y is continuous on $(a_i,\tau_i^{(1)})$, we have that, in fact, $y(\tau)\to l$ as $\tau\nearrow\tau_i^{(1)}$. If $\tau_i^{(1)}=1$, then $(l,\tau_i^{(1)})$ cannot be non-hyperbolic and so is a hyperbolic stable zero, and, therefore, the domain of y can be extended to include 1 itself.

Thus, if $\tau_i^{(1)}=b_i$, then we have already defined y on the whole of the interior of C_i relative to $[0,1]$. So, now suppose that $\tau_i^{(1)}\neq b_i$. Note that $(l,\tau_i^{(1)})$ must be non-hyperbolic, otherwise by Remark 9(A), $\tau_i^{(1)}$ would not be the supremum of the set of times up to which we can uniquely extend y. Since $(l,\tau_i^{(1)})$ is non-hyperbolic, Remark 9(B) implies that there are no zeros (θ,τ) close to $(l,\tau_i^{(1)})$ with $\tau>\tau_i^{(1)}$, and, therefore, $\mathbb{S}^1\times\{\tau_i^{(1)}\}$ must contain zeros other than just $(l,\tau_i^{(1)})$. So, let $(\theta_1,\tau_i^{(1)})$ be the zero for which there is a fast connection from $(l,\tau_i^{(1)})$ to $(\theta_1,\tau_i^{(1)})$. Since l is a local extremum of $F(\cdot,\tau_i^{(1)})$, it is clear that θ_1 is unique. Since $(l,\tau_i^{(1)})$ is non-hyperbolic, $(\theta_1,\tau_i^{(1)})$ cannot also be non-hyperbolic and, therefore, $(\theta_1,\tau_i^{(1)})$ must be hyperbolic stable. Hence, Remark 9(A) gives that for some $\delta_1>0$, there is a unique curve of the stable slow manifold over $(\tau_i^{(1)},\tau_i^{(1)}+\delta_1)$ whose right-sided limit at $\tau_i^{(1)}$ coincides with θ_1. So, extend the domain of y to include the open interval $(\tau_i^{(1)},\tau_i^{(1)}+\delta_1)$, with $y|_{(\tau_i^{(1)},\tau_i^{(1)}+\delta_1)}$ being the unique curve of the stable slow manifold over $(\tau_i^{(1)},\tau_i^{(1)}+\delta_1)$ for which $y(\tau)\to\theta_1$ as $\tau\searrow\tau_i^{(1)}$. Now, define $\tau_i^{(2)}$ as the maximum of the set of all $\tau>\tau_i^{(1)}$ for which there is a unique curve of the stable slow manifold over $(\tau_i^{(1)},\tau)$ agreeing with y on $(\tau_i^{(1)},\tau_i^{(1)}+\delta_1)$, and extend y to include $(\tau_i^{(1)},\tau_i^{(2)})$. We can then treat $\tau_i^{(2)}$ the same way we treated $\tau_i^{(1)}$ earlier; and continuing the procedure as necessary, we obtain an increasing list of terms $\tau_i^{(1)},\tau_i^{(2)},\tau_i^{(3)},\dots$ that terminates at the term $\tau_i^{(m_i)}$ for which $\tau_i^{(m_i)}=b_i$. Note that this termination must take place due to the fact that there are only finitely many τ-values for which $\mathbb{S}^1\times\{\tau\}$ contains a non-hyperbolic zero.

Thus, overall, we have defined y on the set

$$U=(C\cap\{0,1\})\cup\bigcup_{i=1}^{n}\bigcup_{j=0}^{m_i-1}(\tau_i^{(j)},\tau_i^{(j+1)}),$$

where $\tau_i^{(0)}=a_i$. By construction, if $0\notin C$, then (U,y) is the unique complete curve of the stable slow manifold, and if $0\in C$, then for our given θ_0, (U, y) is the unique complete curve of the stable slow manifold fulfilling $y(0) = \theta_0$.

C. When F has no zeros

In this section, we will prove Proposition 26 and Theorem 24(A) as well as obtaining a corollary of Theorem 24(A) that will be needed later for the proof of Theorem 24(C).

For each $n \geq 1$, let $\mathfrak{e}_n \colon \mathbb{R} \to \mathbb{R}$ be the continuous function for which $\mathfrak{e}_n(x)x^n$ is the remainder in the order-$(n-1)$ Taylor expansion of e^x; in other words, $\mathfrak{e}_n(x) = \sum_{i=0}^{\infty} \frac{x^i}{(i+n)!}$. Note, in particular, that $\mathfrak{e}_n$ is strictly increasing on $[0, \infty)$. It is easy to check that for any $c, t \in \mathbb{R}$,

$$\int_0^t s e^{c(t-s)}\, ds = \mathfrak{e}_2(ct)t^2$$

and

$$\int_0^t \mathfrak{e}_n(cs)s^n\, ds = \mathfrak{e}_{n+1}(ct)t^{n+1}.$$

If F has no zeros, then we will work with all the notations introduced in Proposition 26, and also let

$$L = \min_{(\theta,\tau)\in\mathbb{S}^1\times[0,1]} |F(\theta,\tau)|.$$

Proof of Proposition 26. Let τ^* be a global minimizer of $\tau \mapsto m_1(\tau)k(\tau)$. Given $\varepsilon > 0$, define an integer $n \geq 1$ and times

$$0 = t_1 < \cdots < t_n$$

such that

- $t_{i+1} = t_i + k(\varepsilon t_i)$ for each $1 \leq i < n$, and
- $\frac{\tau^*}{\varepsilon} \in [t_n, t_n + k(\varepsilon t_n))$;

and then define an integer $\tilde{n} \geq 1$ and times

$$\tfrac{1}{\varepsilon} = \tilde{t}_1 > \cdots > \tilde{t}_{\tilde{n}}$$

such that

- $\tilde{t}_{i+1} = \tilde{t}_i - k(\varepsilon\tilde{t}_i)$ for each $1 \leq i < \tilde{n}$, and
- $t_n \in (\tilde{t}_{\tilde{n}} - k(\varepsilon\tilde{t}_{\tilde{n}}), \tilde{t}_{\tilde{n}}]$.

We will approximate $\int_0^1 r(\tau)\, d\tau$ by Riemann-like sums over the partition of $[0, 1]$ whose set of end points is given by $\{\varepsilon t_i\}_{1\leq i\leq n} \cup \{\varepsilon\tilde{t}_i\}_{1\leq i\leq\tilde{n}}$. Note that the quantities $t_{i+1} - t_i$, $\tilde{t}_i - \tilde{t}_{i+1}$, and $\tilde{t}_{\tilde{n}} - t_n$ are all bounded by the ε-independent constant $K := \max_{\tau\in[0,1]} k(\tau)$, and hence the mesh size of this partition tends to 0 as $\varepsilon \to 0$.

For any continuous function $g \colon [0, 1] \to \mathbb{R}$, define

$$\begin{aligned} g^{\vee i} &= \max_{\tau\in[\varepsilon t_i,\varepsilon t_{i+1}]} g(\tau) \quad \text{for each } 1 \leq i < n, \\ g^{\tilde{\vee} i} &= \max_{\tau\in[\varepsilon\tilde{t}_{i+1},\varepsilon\tilde{t}_i]} g(\tau) \quad \text{for each } 1 \leq i < \tilde{n} \\ g^{\tilde{\vee}\tilde{n}} &= \max_{\tau\in[\varepsilon t_n,\varepsilon\tilde{t}_{\tilde{n}}]} g(\tau). \end{aligned}$$

Let

$$r_i = m_{11}^{\vee i} m_2^{\vee i} k(\varepsilon t_i)^2 \mathfrak{e}_3(m_1^{\vee i} k(\varepsilon t_i)) + \tfrac{1}{2} m_{12}^{\vee i} k(\varepsilon t_i), \tag{36}$$

$$\tilde{r}_i = m_{11}^{\tilde{\vee} i} m_2^{\tilde{\vee} i} k(\varepsilon\tilde{t}_i)^2 \mathfrak{e}_3(m_1^{\tilde{\vee} i} k(\varepsilon\tilde{t}_i)) + \tfrac{1}{2} m_{12}^{\tilde{\vee} i} k(\varepsilon\tilde{t}_i). \tag{37}$$

We have that $r(\varepsilon t_i) \leq r_i \leq r^{\vee i}$ and $r(\varepsilon\tilde{t}_i) \leq \tilde{r}_i \leq r^{\tilde{\vee} i}$; hence,

$$\max\big(\{r_i - r(\varepsilon t_i)\}_{1\leq i<n} \cup \{\tilde{r}_i - r(\varepsilon\tilde{t}_i)\}_{1\leq i<\tilde{n}}\big) \to 0$$

and, therefore,

$$\left(\sum_{i=0}^{n-1} \varepsilon k(\varepsilon t_i) r_i\right) + \left(\sum_{i=0}^{\tilde{n}-1} \varepsilon k(\varepsilon\tilde{t}_i)\tilde{r}_i\right) \to \int_0^1 r(\tau)\, d\tau, \tag{38}$$

as $\varepsilon \to 0$. We also have that

$$|\tfrac{\tau^*}{\varepsilon} - \tilde{t}_{\tilde{n}}| < \max(k(\varepsilon t_n), k(\varepsilon\tilde{t}_{\tilde{n}})) \leq K$$

and hence

$$m_1^{\tilde{\vee}\tilde{n}} k(\varepsilon\tilde{t}_{\tilde{n}}) \to m_1(\tau^*)k(\tau^*), \tag{39}$$

as $\varepsilon \to 0$. So, to prove the desired result, by Eqs. (38) and (39), it is sufficient to show that

$$|\bar{\lambda}^{\varepsilon,F}_{0,\frac{1}{\varepsilon}}(\theta_0)| \leq \left(\sum_{i=0}^{n-1} \varepsilon k(\varepsilon t_i) r_i\right) + \left(\sum_{i=0}^{\tilde{n}-1} \varepsilon k(\varepsilon\tilde{t}_i)\tilde{r}_i\right) + m_1^{\tilde{\vee}\tilde{n}} k(\varepsilon\tilde{t}_{\tilde{n}}) \tag{40}$$

for all $\theta_0 \in \mathbb{S}^1$.

Fix $\theta_0 \in \mathbb{S}^1$. Let $\theta \colon \left[0, \frac{1}{\varepsilon}\right] \to \mathbb{S}^1$ be the solution of (12) starting at $\theta(0) = \theta_0$, and let $\hat{\theta} \colon \left[0, \frac{1}{\varepsilon}\right] \to \mathbb{R}$ be a lift of θ. For each $1 \leq i < n$, let $\psi_i \colon \mathbb{R} \to \mathbb{S}^1$ be the solution of the autonomous differential equation

$$\dot{\psi}_i = F(\psi_i, \varepsilon t_i) \tag{41}$$

for which $\psi_i(t_i) = \theta(t_i)$, and let $\hat{\psi}_i \colon \mathbb{R} \to \mathbb{R}$ be a lift of ψ_i with $\hat{\psi}_i(t_i) = \hat{\theta}(t_i)$. For all $t \in [t_i, t_{i+1}]$, we have

$$\begin{aligned} |\hat{\theta}(t) - \hat{\psi}_i(t)| &\leq \int_{t_i}^t |\dot{\hat{\theta}}(s) - \dot{\hat{\psi}}_i(s)|\, ds \\ &= \int_{t_i}^t |F(\theta(s), \varepsilon s) - F(\psi_i(s), \varepsilon t_i)|\, ds \\ &\leq \int_{t_i}^t |F(\theta(s), \varepsilon s) - F(\psi_i(s), \varepsilon s)| \\ &\quad + |F(\psi_i(s), \varepsilon s) - F(\psi_i(s), \varepsilon t_i)|\, ds \\ &\leq \int_{t_i}^t m_1^{\vee i}|\hat{\theta}(s) - \hat{\psi}_i(s)| + m_2^{\vee i}\varepsilon(s - t_i)\, ds \end{aligned}$$

and so a suitable version of Grönwall's inequality (Ref. 50, Corollary 2) gives that

$$\begin{aligned} |\hat{\theta}(t) - \hat{\psi}_i(t)| &\leq \int_{t_i}^t m_2^{\vee i}\varepsilon(s - t_i)\, e^{m_1^{\vee i}(t-s)}\, ds \\ &= \varepsilon m_2^{\vee i}\mathfrak{e}_2(m_1^{\vee i}(t - t_i))(t - t_i)^2. \end{aligned}$$

Now, all solutions of (41) are $k(\varepsilon t_i)$-periodic, and so Eq. (33) applied to the τ-independent function $F(\theta, \varepsilon t_i)$ gives

$$\int_{t_i}^{t_{i+1}} \tfrac{\partial F}{\partial\theta}(\psi_i(t), \varepsilon t_i)\, dt = 0.$$

Hence,

$$
\begin{aligned}
&|\bar{\lambda}^{\varepsilon,F}_{t_i,t_{i+1}}(\theta_0)| \\
&= \left|\bar{\lambda}^{\varepsilon,F}_{t_i,t_{i+1}}(\theta_0) - \int_{t_i}^{t_{i+1}} \tfrac{\partial F}{\partial\theta}(\psi_i(t),\varepsilon t_i)\,dt\right| \\
&= \left|\int_{t_i}^{t_{i+1}} \tfrac{\partial F}{\partial\theta}(\theta(t),\varepsilon t) - \tfrac{\partial F}{\partial\theta}(\psi_i(t),\varepsilon t_i)\,dt\right| \\
&\le \int_{t_i}^{t_{i+1}} \left|\tfrac{\partial F}{\partial\theta}(\theta(t),\varepsilon t) - \tfrac{\partial F}{\partial\theta}(\psi_i(t),\varepsilon t)\right| \\
&\quad + \left|\tfrac{\partial F}{\partial\theta}(\psi_i(t),\varepsilon t) - \tfrac{\partial F}{\partial\theta}(\psi_i(t),\varepsilon t_i)\right|\,dt \\
&\le \int_{t_i}^{t_{i+1}} m_{11}^{\vee i}|\hat{\theta}(t)-\hat{\psi}(t)| + m_{12}^{\vee i}\varepsilon(t-t_i)\,dt \\
&\le \int_{t_i}^{t_{i+1}} m_{11}^{\vee i}\varepsilon m_2^{\vee i}\mathfrak{e}_2(m_1^{\vee i}(t-t_i))(t-t_i)^2 + m_{12}^{\vee i}\varepsilon(t-t_i)\,dt \\
&= \varepsilon\left(m_{11}^{\vee i}m_2^{\vee i}\mathfrak{e}_3(m_1^{\vee i}k(\varepsilon t_i))k(\varepsilon t_i)^3 + \tfrac{1}{2}m_{12}^{\vee i}k(\varepsilon t_i)^2\right) \\
&= \varepsilon k(\varepsilon t_i)r_i.
\end{aligned}
$$

By the same reasoning applied to the time-reversal of (12) (i.e., applied to $F_{1,0}$ in place of F), we likewise have that for all $1 \le i < \tilde{n}$,

$$|\bar{\lambda}^{\varepsilon,F}_{\tilde{t}_{i+1},\tilde{t}_i}(\theta_0)| \le \varepsilon k(\varepsilon\tilde{t}_i)\tilde{r}_i.$$

Finally,

$$|\bar{\lambda}^{\varepsilon,F}_{t_n,\tilde{t}_{\tilde{n}}}(\theta_0)| \le \int_{t_n}^{\tilde{t}_{\tilde{n}}} \left|\tfrac{\partial F}{\partial\theta}(\theta(t),\varepsilon t)\right| \le m_1^{\tilde{\vee}\tilde{n}}k(\varepsilon\tilde{t}_{\tilde{n}}).$$

Combining our bounds on $|\bar{\lambda}^{\varepsilon,F}_{t_i,t_{i+1}}(\theta_0)|$, $|\bar{\lambda}^{\varepsilon,F}_{\tilde{t}_{i+1},\tilde{t}_i}(\theta_0)|$, and $|\bar{\lambda}^{\varepsilon,F}_{t_n,\tilde{t}_{\tilde{n}}}(\theta_0)|$ yields the required Eq. (40). □

Proof of Theorem 24(A). Continuing with the notations in the proof of Proposition 26, and letting

$$R = M_{11}M_2K^2\mathfrak{e}_3(M_1K) + \tfrac{1}{2}M_{12}K,$$

we have that $r_i \le R$ for all $1 \le i < n$ and $\tilde{r}_i \le R$ for all $1 \le i < \tilde{n}$, and we also have that

$$\left(\sum_{i=0}^{n-1} k(\varepsilon t_i)\right) + \left(\sum_{i=0}^{\tilde{n}-1} k(\varepsilon\tilde{t}_i)\right) = \tfrac{1}{\varepsilon} - (\tilde{t}_{\tilde{n}} - t_n) \le \tfrac{1}{\varepsilon}.$$

Hence,

$$\left(\sum_{i=0}^{n-1} \varepsilon k(\varepsilon t_i)r_i\right) + \left(\sum_{i=0}^{\tilde{n}-1} \varepsilon k(\varepsilon\tilde{t}_i)\tilde{r}_i\right) \le R$$

and hence, by Eq. (40),

$$|\bar{\lambda}^{\varepsilon,F}_{0,\frac{1}{\varepsilon}}(\theta_0)| \le R + M_1K =: C[F]$$

for all $\varepsilon > 0$ and $\theta_0 \in \mathbb{S}^1$. Now, it is easy to see that $C[F_{\sigma_1,\sigma_2}] \le C[F]$, for any $\sigma_1,\sigma_2 \in [0,1]$. Hence,

$$|\bar{\lambda}^{\varepsilon,F}_{s,t}(\theta_0)| \le C[F_{\varepsilon s,\varepsilon t}] \le C[F], \tag{42}$$

for all $s,t \in [0,\frac{1}{\varepsilon}]$. So, by Proposition 6, F is neutrally stable with constant $c = e^{C[F]}$. It is clear that $C[\cdot]$ is locally bounded (in fact, continuous) with respect to the norm $\|\cdot\|_{\mathscr{F}}$, and hence F is robustly neutrally stable. □

The following corollary will play a similar role in the proof of Theorem 24(C) to the role played by Lemma 2 of Ref. 17 in the main results of Ref. 17.

Corollary 45. *Suppose F has no zeros, and let*

$$\mathrm{sgn}(F) = \begin{cases} 1, & F > 0 \text{ everywhere}, \\ -1, & F < 0 \text{ everywhere}. \end{cases}$$

Then, for all $\varepsilon > 0$ and $\theta_0 \in \mathbb{S}^1$,

$$\mathrm{sgn}(F)\tfrac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon},\theta_0,0,1\right) \ge c^{-1}L,$$

where $c = e^{C[F]}$ is as in the proof of Theorem 24(A).

Proof. Follows immediately from Eqs. (42) and (34). □

For the sake of completeness, in Appendix A, we give bounds on the limiting range of $\frac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\frac{1}{\varepsilon},\theta_0,0,1\right)$ as $\varepsilon \to 0$.

Now, as an immediate consequence of Corollary 45, we obtain that if F has no zeros, then

$$\mathrm{sgn}(F)\Delta\Phi\left(\tfrac{1}{\varepsilon},\theta_0\right) \ge \tfrac{1}{\varepsilon}c^{-1}L, \tag{43}$$

for all $\varepsilon > 0$ and $\theta_0 \in \mathbb{S}^1$. [In fact, as one would intuitively expect, it is not hard to show by elementary means that

$$\varepsilon\Delta\Phi\left(\tfrac{1}{\varepsilon},\theta_0\right) \to \mathrm{sgn}(F)\int_0^1 \frac{2\pi}{k(\tau)}\,d\tau$$

uniformly across $\theta_0 \in \mathbb{S}^1$ as $\varepsilon \to 0$. Further results on this convergence are given in Ref. 17 between Eqs. (4.20) and (4.22).]

D. More on tracking

We first give the result that provides the connection between tracking and stability in Theorem 24(B) and (C); it is a consequence of Theorem 24(A) or Proposition 26. In the following, $\mathrm{Leb}(\cdot)$ denotes the Lebesgue measure on $[0,1]$.

Lemma 46. *Suppose F is generic, and let (U,y) be a complete curve of the stable manifold, with Λ_{ad} being the corresponding adiabatic Lyapunov exponent. Let $S \subset U$ be a Borel set. For all $\eta > 0$, there exist $\delta_0,\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and $\delta \in (0,\delta_0)$, if $\theta(\cdot)$ is a solution of (12) that δ-tracks $(S,y|_S)$ over $\left[0,\frac{1}{\varepsilon}\right]$, then [writing $\theta(0) =: \theta_0$],*

$$|\lambda^{\varepsilon,F}_{0,\frac{1}{\varepsilon}}(\theta_0) - \Lambda_{\mathrm{ad}}| \le 2M_1\mathrm{Leb}(U\setminus S) + \eta.$$

Proof. Essentially by definition,

$$\lambda^{\varepsilon,F}_{0,\frac{1}{\varepsilon}}(\theta_0) - \Lambda_{\mathrm{ad}} = \varepsilon\int_0^{\frac{1}{\varepsilon}} \tfrac{\partial F}{\partial\theta}(\theta(s),\varepsilon s) - \mathfrak{Y}(s)\,ds,$$

with

$$\mathfrak{Y}(s) := \begin{cases} \tfrac{\partial F}{\partial\theta}(y(\varepsilon s),\varepsilon s), & \varepsilon s \in U, \\ 0, & \varepsilon s \notin U. \end{cases}$$

Now, first observe that for any solution $\theta(\cdot)$ of (12) (with any $\varepsilon > 0$), we have

$$
\begin{aligned}
&\varepsilon\int_{\{t:\varepsilon t\in U\setminus S\}} \left|\tfrac{\partial F}{\partial\theta}(\theta(s),\varepsilon s) - \tfrac{\partial F}{\partial\theta}(y(\varepsilon s),\varepsilon s)\right|\,ds \\
&\le 2M_1\mathrm{Leb}(U\setminus S).
\end{aligned} \tag{44}
$$

Now, fix $\eta > 0$. Taking $\delta_0 = \frac{\eta}{3M_{11}}$, we have that for any $\varepsilon > 0$ and $\delta \in (0, \delta_0)$, if $\theta(\cdot)$ is a solution of (12) that δ-tracks $(S, y|_S)$ over $[0, \frac{1}{\varepsilon}]$, then

$$\varepsilon \int_{\{t:\varepsilon t\in S\}} \left| \tfrac{\partial F}{\partial\theta}(\theta(s), \varepsilon s) - \tfrac{\partial F}{\partial\theta}(y(\varepsilon s), \varepsilon s)\right| ds \\ \le \delta M_{11}\mathrm{Leb}(S) \le \tfrac{\eta}{3}\mathrm{Leb}(S). \tag{45}$$

Now, let $W \subset [0,1] \setminus \bar{U}$ be a disjoint union of finitely many intervals $[w_i^-, w_i^+]$, $i = 1, \ldots, n$, such that

$$\mathrm{Leb}([0,1] \setminus (U \cup W)) < \frac{\eta}{3M_1}.$$

So, for any solution $\theta(\cdot)$ of (12) (with any $\varepsilon > 0$), we have

$$\varepsilon \int_{\{t:\varepsilon t\in[0,1]\setminus(U\cup W)\}} \left| \tfrac{\partial F}{\partial\theta}(\theta(s), \varepsilon s)\right| ds \le \tfrac{\eta}{3}. \tag{46}$$

For each $1 \le i \le n$, $F_{w_i^-, w_i^+}$ has no zeros; and so on the basis of Theorem 24(A) or Proposition 26, let $\varepsilon_0 > 0$ be such that for all $\varepsilon \in (0, \varepsilon_0)$ and $1 \le i \le n$, every solution $\theta(\cdot)$ of (12) has

$$\varepsilon \left| \int_{\frac{w_i^-}{\varepsilon}}^{\frac{w_i^+}{\varepsilon}} \tfrac{\partial F}{\partial\theta}(\theta(s), \varepsilon s)\, ds \right| \le \tfrac{\eta}{3n}. \tag{47}$$

Combining the four Eqs. (44)–(47) yields the desired result. □

Let us now introduce a few further tracking-related definitions. In the following, we assume that we have a set $V \subset [0,1]$, a function $q\colon V \to \mathbb{S}^1$, and a point $p \in \mathbb{S}^1$.

Definition 47. Fix $\varepsilon > 0$. Given $\Delta, \delta > 0$, we say that (12) *exhibits* (Δ, δ)-*tracking of* (V, q) *away from* p if every solution $\theta(\cdot)$ of (12) with $d(\theta(0), p) \ge \frac{\Delta}{2}$ δ-tracks (V, q).

Definition 48. We say that F *exhibits strict tracking of* (V, q) *away from* p if given any $\Delta, \delta > 0$, for sufficiently small ε, (12) exhibits (Δ, δ)-tracking of (V, q) away from p.

Definition 49. Assume that V is σ-compact. We say that F *exhibits tracking of* (V, q) *away from* p if for every compact subset S of $V \setminus \{0\}$, F exhibits strict tracking of $(S, q|_S)$ away from p.

E. When $\bar{U}$ is connected and $0 \in U$

Proposition 50. *Suppose F is generic and not initially multistable, and that $\bar{U}$ is connected and $0 \in U$. Let $(z_0, 0)$ be the unique hyperbolic unstable zero of F in $\mathbb{S}^1 \times \{0\}$. Then, F exhibits tracking of (U, y) away from z_0.*

In other words, the conclusion of Proposition 50 states that by taking ε sufficiently small, we can guarantee that solutions $\theta(t)$ that do not start too close to z_0 will be close to $y(\varepsilon t)$ at all times $t \in \frac{1}{\varepsilon}S$, where $S \subset U$ is bounded away from $\{0\} \cup \partial U$.

This is fairly obvious given the definition of (U, y). Unsurprisingly, to write out a rigorous proof would involve no conceptual difficulties or subtleties but would, nonetheless, be technically tedious. Therefore, we will omit a proof.

Combining Proposition 50 with Lemma 46 gives the following immediate consequence:

Proposition 51. *In the setting of Proposition 50, F is exponentially stable away from z_0 with rate Λ_{ad}.*

Proof. Fix $\eta, \Delta > 0$, let $S \subset U \setminus \{0\}$ be a compact set with $2M_1\mathrm{Leb}(U \setminus S) < \frac{\eta}{2}$, let δ_0, ε_0 be the values given by Lemma 46 applied to S with $\frac{\eta}{2}$ in place of η, and take any $\delta \in (0, \delta_0)$. On the basis of Proposition 50, suppose $\varepsilon < \varepsilon_0$ is sufficiently small that (12) exhibits (Δ, δ)-tracking of $(S, y|_S)$ away from z_0. Then, every solution $\theta(\cdot)$ of (12) with $\theta(0) \notin B_{\frac{\Delta}{2}}(z_0)$ has $|\lambda^{\varepsilon,F}_{0,\frac{1}{\varepsilon}}(\theta(0)) - \Lambda_{\mathrm{ad}}| \le \eta$. □

F. Proof of Theorem 24(B) for $0 \notin U$

Assume that F is generic, $\mathbb{S}^1 \times \{0\}$ contains no zeros, and $\bar{U}$ is connected.

To show that F exhibits tracking of (U, y), we need to show that for any compact $S \subset U$ and any $\Delta, \delta > 0$, if ε is sufficiently small, then there is an arc P_ε of length less of Δ such that every solution of (12) starting outside of P_ε δ-tracks $(S, y|_S)$. Fix S, Δ and δ. Let $\tau^{(0)} = \inf U$ and let $\sigma_0 = \min S$. We have that $\mathbb{S}^1 \times \{\tau^{(0)}\}$ contains exactly one zero of F, namely, a non-hyperbolic zero; so, on the basis of Remark 9(B), choose a value $\tilde{\tau} \in (\tau^{(0)}, \sigma_0)$ such that the only zeros in $\mathbb{S}^1 \times \{\tilde{\tau}\}$ are $(y(\tilde{\tau}), \tilde{\tau})$ and a hyperbolic unstable zero $(z(\tilde{\tau}), \tilde{\tau})$. Note that $F_{\tilde{\tau},0}$ is generic and not initially multistable, and the corresponding complete curve $(U_{\tilde{\tau},0}, y_{\tilde{\tau},0})$ of the stable slow manifold has

$$\bar{U}_{\tilde{\tau},0} = \left[0, \tfrac{\tilde{\tau}-\tau^{(0)}}{\tilde{\tau}}\right],$$

$$y_{\tilde{\tau},0}(0) = z(\tilde{\tau}).$$

Take any $\tilde{\Delta} < 2\, d(y(\tilde{\tau}), z(\tilde{\tau}))$ and let $\tilde{P} = B_{\frac{\tilde{\Delta}}{2}}(z(\tilde{\tau}))$. We have that

(a) by Proposition 50 applied to $F_{\tilde{\tau},1}$, if ε is sufficiently small then every solution $\theta(\cdot)$ of (12) with $\theta\left(\frac{\tilde{\tau}}{\varepsilon}\right) \notin \tilde{P}$ δ-tracks $(S, y|_S)$, and

(b) by Proposition 51 applied to $F_{\tilde{\tau},0}$, since $y(\tilde{\tau})$ has a neighborhood that does not intersect $\tilde{P}$, the length of the arc

$$P_\varepsilon := \{\theta(0) \,:\, \theta(\cdot) \text{ solves (12)}, \theta\left(\tfrac{\tilde{\tau}}{\varepsilon}\right) \in \tilde{P}\} \tag{48}$$

tends to 0 exponentially as $\varepsilon \to 0$; and so $\mathrm{length}(P_\varepsilon) < \Delta$ for sufficiently small ε.

Combining (a) and (b) gives the desired result.

Having shown that F exhibits tracking of (U, y), combining this with Lemma 46 gives that F is exponentially stable with rate Λ_{ad}, by exactly the same reasoning as in the proof of Proposition 51.

G. Proof of Proposition 25

We start with the following general result.

Proposition 52. *Suppose $F \in \mathscr{F}$ is such that for some $\tau_* \in (0, 1)$,*

- *F has no zeros in $\mathbb{S}^1 \times (\tau_*, 1]$;*
- *for all $\tau \in [0, \tau_*)$, $\mathbb{S}^1 \times \{\tau\}$ contains exactly two zeros of F, namely, a hyperbolic stable zero $(y(\tau), \tau)$ and a hyperbolic unstable zero $(z(\tau), \tau)$; and*
- *letting*

$$\mathfrak{s}_F = \begin{cases} 1, & F > 0 \text{ on } \mathbb{S}^1 \times (\tau_*, 1], \\ -1, & F < 0 \text{ on } \mathbb{S}^1 \times (\tau_*, 1], \end{cases}$$

we have $\mathfrak{s}_F \frac{\partial F}{\partial\tau}(y(\tau), \tau) > 0$ and $\mathfrak{s}_F \frac{\partial F}{\partial\tau}(z(\tau), \tau) > 0$ for all $\tau \in [0, \tau_*)$.

Then,

$$\min_{\theta_0\in\mathbb{S}^1} \mathfrak{s}_F\Delta\Phi\left(\tfrac{1}{\varepsilon},\theta_0\right) \to \infty \quad \textit{as}\ \varepsilon \to 0.$$

Proof. First, take θ_0 with $\mathfrak{s}_F F(\theta_0,0) > 0$. Let $\theta(\cdot)$ be the solution of (12) with $\theta(0)=\theta_0$, and let $\hat\theta(\cdot)$ be a lift of $\theta(\cdot)$. By the implicit function theorem, on the whole of $[0,\tau_*)$, the derivative of $\tau \mapsto y(\tau)$ has sign $\mathfrak{s}_F$ and the derivative of $\tau \mapsto z(\tau)$ has sign $-\mathfrak{s}_F$. So, it is clear that $\theta(t)$ cannot cross past either $y(\varepsilon t)$ or $z(\varepsilon t)$ during $t \in [0,\frac{\tau_*}{\varepsilon})$. Therefore, $\mathfrak{s}_F\dot\theta(t) = \mathfrak{s}_F F(\theta(t),\varepsilon t) > 0$ for all $t \in [0,\frac{\tau_*}{\varepsilon})$, and so

$$\mathfrak{s}_F\left(\hat\theta\left(\tfrac{\tau_*}{\varepsilon}\right) - \hat\theta(0)\right) > 0.$$

It follows that if we now let θ_0 be an arbitrary point in $\mathbb{S}^1$, then

$$\mathfrak{s}_F\left(\hat\theta\left(\tfrac{\tau_*}{\varepsilon}\right) - \hat\theta(0)\right) > -2\pi.$$

Furthermore, obviously $\mathfrak{s}_F F(\theta(t),\varepsilon t) > 0$ for all $t \in \left(\frac{\tau_*}{\varepsilon},\frac{1}{\varepsilon}\right]$. And, taking an arbitrary $\tau_{**} \in (\tau_*,1)$, we have that $\mu := \min_{(\theta,\tau)\in\mathbb{S}^1\times[\tau_{**},1]} \mathfrak{s}_F F(\theta,\tau) > 0$. Hence,

$$\min_{\theta_0\in\mathbb{S}^1} \mathfrak{s}_F\Delta\Phi\left(\tfrac{1}{\varepsilon},\theta_0\right) > -2\pi + \tfrac{\mu(1-\tau_{**})}{\varepsilon}.$$

□

Now, suppose F is generic and not initially multistable, and $\bar U$ is not connected. Let $[\tau_0^-,\tau_0^+]$ be the first connected component of $\bar U$ and let τ_1 be the lower end point of the second connected component of $\bar U$. Fix an arbitrary $\sigma \in (\tau_0^+,\tau_1)$. On the basis of Remark 9(B) applied to the non-hyperbolic zero in $\mathbb{S}^1\times\{\tau_0^+\}$ and in $\mathbb{S}^1\times\{\tau_1\}$, we can find $\tilde\tau_0 < \tau_0^+$ and $\tilde\tau_1 > \tau_1$ such that $F_{\tilde\tau_0,\sigma}$ and $F_{\tilde\tau_1,\sigma}$ each fulfill the conditions of Proposition 52. For each $\tau \in (\tau_1,\tilde\tau_1]$, let $(z(\tau),\tau)$ be the hyperbolic unstable zero in $\mathbb{S}^1\times\{\tau\}$. Let

$$\Lambda_0 = \int_{\tau_0^-}^{\tau_0^+} \tfrac{\partial F}{\partial\theta}(y(\tau),\tau)\,d\tau$$

and fix $\tau' \in (\tau_1,\tilde\tau_1]$ sufficiently close to τ_1 that

$$\tilde\Lambda := \int_{\tau_1}^{\tau'} \tfrac{\partial F}{\partial\theta}(z(\tau),\tau)\,d\tau < -\Lambda_0.$$

Fix an arbitrary $\eta > 0$ sufficiently small that $\tilde\Lambda + \eta < -\Lambda_0 - \eta$.

Now, we will show that for every compact $S \subset (\tau_1,\tau']$ and every $\Delta,\delta > 0$, there are intervals of ε-values reaching arbitrarily close to 0 for which one can find an arc P_ε of length less than Δ such that every solution $\theta(\cdot)$ of (12) starting outside P_ε δ-tracks $(S,z|_S)$. Without loss of generality, take S of the form $S=[\tau'',\tau']$ for some $\tau'' \in (\tau_1,\tau')$. Without loss of generality, take $\delta < d(y(\tau''),z(\tau''))$, and let $Q = B_\delta(z(\tau'))$. Since $F_{\tilde\tau_1,\sigma}$ fulfills the conditions of Proposition 52, we have, in particular, that $\delta < d(y(\tau'),z(\tau'))$. Fix any $\Delta > 0$. Also define the positive quantities

$$\upsilon := \min_{\tau\in S}\min(F(z(\tau)+\delta,\tau),-F(z(\tau)-\delta,\tau)),$$

$$\nu := \max_{\tau\in S}|z'(\tau)|,$$

where z' denotes the derivative of z. Using Theorem 24(B) applied to $F_{0,\sigma}$ and Proposition 51 applied to $F_{\tau',\sigma}$ [where in the latter case, we note that $y(\tau')$ has a neighborhood not intersecting the arc Q], we can find $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the following statements hold:

(1) There is an arc P_ε of length less than Δ such that, writing

$$\tilde J(\varepsilon) = \{\theta\left(\tfrac{\sigma}{\varepsilon}\right) : \theta(\cdot)\text{ solves (12)}, \theta(0)\notin P_\varepsilon\},$$

we have

$$\text{length}(\tilde J(\varepsilon)) < 2\pi\cdot e^{\frac{\tilde\Lambda+\eta}{\varepsilon}}.$$

(2) Writing

$$\tilde{\tilde J}(\varepsilon) = \{\theta\left(\tfrac{\sigma}{\varepsilon}\right) : \theta(\cdot)\text{ solves (12)}, \theta\left(\tfrac{\tau'}{\varepsilon}\right)\in Q\},$$

we have

$$\text{length}(\tilde{\tilde J}(\varepsilon)) > 2\delta\cdot e^{\frac{-\Lambda_0-\eta}{\varepsilon}}.$$

(3) We have $\varepsilon\nu < \upsilon$; and therefore, for any solution $\theta(\cdot)$ of (12), if $\theta(\frac{\tau'}{\varepsilon}) \in Q$, then $d(\theta(t),z(\varepsilon t)) < \delta$ for all t with $\varepsilon t \in S$.

Note that $\tilde{\tilde J}(\varepsilon)$ depends continuously on ε. One can also take P_ε and hence $\tilde J(\varepsilon)$ to depend continuously on ε: if $0 \in U$, then by Proposition 51 P_ε can be taken independent of ε, and if $0 \notin U$ then one can see from the proof of Theorem 24(B) [see, in particular, Eq. (48)] that P_ε can be taken to depend continuously on ε. By Proposition 52 applied to $F_{\tau',\sigma}$, $\tilde{\tilde J}(\varepsilon)$ moves unboundedly round the circle as $\varepsilon \to 0$. Also, by the tracking statement in Theorem 24(B) applied to $F_{0,\tilde\tau_0}$, it is clear that the set

$$\begin{aligned}\tilde J_0(\varepsilon) &:= \{\theta\left(\tfrac{\tilde\tau_0}{\varepsilon}\right) : \theta(\cdot)\text{ solves (12)}, \theta(0)\notin P_\varepsilon\}\\ &= \{\theta\left(\tfrac{\tilde\tau_0}{\varepsilon}\right) : \theta(\cdot)\text{ solves (12)}, \theta\left(\tfrac{\sigma}{\varepsilon}\right)\in\tilde J(\varepsilon)\}\end{aligned}$$

remains within an ε-independent proper subset of $\mathbb{S}^1$ for all sufficiently small ε, and, therefore, applying Proposition 52 to $F_{\tilde\tau_0,\sigma}$ gives that $\tilde J(\varepsilon)$ moves unboundedly round the circle as $\varepsilon \to 0$ in the opposite direction to $\tilde{\tilde J}(\varepsilon)$. However, by points (1) and (2), we also have that $\text{length}(\tilde J(\varepsilon)) < \text{length}(\tilde{\tilde J}(\varepsilon))$ for all sufficiently small ε. Hence, there are intervals of ε-values arbitrarily close to 0 for which $\tilde J(\varepsilon) \subset \tilde{\tilde J}(\varepsilon)$. For each such ε, every solution $\theta(\cdot)$ of (12) starting outside P_ε has $\theta(\frac{\tau'}{\varepsilon}) \in Q$ and hence, by point (3), $d(\theta(t),z(\varepsilon t)) < \delta$ for all t with $\varepsilon t \in S$.

H. Proof of Theorem 24(C)

We start with the following general result, which is a consequence of Corollary 45.

Proposition 53. *In the setting of Proposition 52, every $\theta_0 \in \mathbb{S}^1$ with $\mathfrak{s}_F F(\theta_0,0) > 0$ has*

$$\inf_{\varepsilon>0} \mathfrak{s}_F \tfrac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon},\theta_0,0,1\right) > 0.$$

Proof. Take an arbitrary $\tau_{**} \in (\tau_*,1)$. By Eq. (34), for any $\varepsilon > 0$ and $\theta_0 \in \mathbb{S}^1$, letting $\theta(\cdot)$ be the solution of (12) with $\theta(0)=\theta_0$, we have

$$\frac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon},\theta_0,0,1\right)$$

$$= \varepsilon\int_0^{\frac{1}{\varepsilon}} e^{\bar{\lambda}^{\varepsilon,F}_{u,\varepsilon^{-1}}(\theta_0)}F(\theta(u),\varepsilon u)\,du$$

$$= \varepsilon\int_0^{\frac{\tau_{**}}{\varepsilon}} e^{\bar{\lambda}^{\varepsilon,F}_{u,\varepsilon^{-1}}(\theta_0)}F(\theta(u),\varepsilon u)\,du + \frac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon},\theta(\tfrac{\tau_{**}}{\varepsilon}),\tau_{**},1\right).$$

By Corollary 45 applied to $F_{\tau_{**},1}$, we have that

$$\inf_{\varepsilon>0,\theta_{**}\in\mathbb{S}^1}\mathfrak{s}_F\frac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon},\theta_{**},\tau_{**},1\right)>0;$$

and as in the proof of Proposition 52, we have that if $\mathfrak{s}_F F(\theta_0,0)>0$, then $\mathfrak{s}_F F(\theta(u),\varepsilon u)\geq 0$ for all u. Hence, the result. □

Now, suppose F is generic and not initially multistable, and $\bar{U}$ has $n\geq 2$ connected components. Let

$$\tau_1^- < \tau_1^+ < \tau_2^- < \tau_2^+ < \cdots < \tau_n^- < \tau_n^+$$

be the end points of the connected components of $\bar{U}$. For each $1\leq i<n$, take an arbitrary value $\tilde{\sigma}_i\in(\tau_i^+,\tau_{i+1}^-)\subset[0,1]\setminus\bar{U}$.

To show that F exhibits tracking of (U,y), we need to show that for any compact $S\subset U\setminus\{0\}$ and any $\Delta,\delta>0$, if ε is sufficiently small, then there is an arc P_ε of length less of Δ such that every solution of (12) starting outside of P_ε δ-tracks $(S,y|_S)$. Fix S, and for each $1\leq i\leq n$, let

$$\sigma_i^- = \min(S\cap[\tau_i^-,\tau_i^+]),$$
$$\sigma_i^+ = \max(S\cap[\tau_i^-,\tau_i^+]).$$

Without loss of generality, assume that for all $1\leq i<n$, σ_i^+ is sufficiently close to τ_i^+ that $F_{\sigma_i^+,\tilde{\sigma}_i}$ fulfills the conditions of Proposition 52. For each $1\leq i<n$, let $(z(\sigma_i^+),\sigma_i^+)$ be the hyperbolic unstable zero in $\mathbb{S}^1\times\{\sigma_i^+\}$. Without loss of generality, let $\delta>0$ be such that for all $1\leq i<n$, $\delta<d(y(\sigma_i^+),z(\sigma_i^+))$. Let $Q_i=B_\delta(y(\sigma_i^+))$ for each $1\leq i<n$. Fix $\Delta>0$, and on the basis of Theorem 24(B) applied to F_{0,σ_1^+}, for all sufficiently small ε, let P_ε be an arc of length less than Δ such that every solution $\theta(\cdot)$ of (12) starting outside P_ε δ-tracks $(S\cap[\tau_1^-,\tau_1^+],y|_{S\cap[\tau_1^-,\tau_1^+]})$.

For each $2\leq i\leq n$, pick a value $\tau_i'\in(\tau_i^-,\sigma_i^-)$ sufficiently close to τ_i^- that $F_{\tau_i',\tilde{\sigma}_{i-1}}$ fulfills the conditions of Proposition 52. For each $2\leq i\leq n$, let $(z(\tau_i'),\tau_i')$ be the hyperbolic unstable zero in $\mathbb{S}^1\times\{\tau_i'\}$, and let $P_{(i)}$ be a neighborhood of $z(\tau_i')$ such $y(\tau_i')\notin\bar{P}_{(i)}$. For each $1\leq i<n$ and each $\varepsilon>0$, let

$$J^-_{(i)}(\varepsilon)=\left\{\theta\left(\tfrac{\tilde{\sigma}_i}{\varepsilon}\right)\ :\ \theta(\cdot)\text{ solves (12)},\theta\left(\tfrac{\sigma_i^+}{\varepsilon}\right)\in Q_i\right\},$$

$$J^+_{(i)}(\varepsilon)=\left\{\theta\left(\tfrac{\tilde{\sigma}_i}{\varepsilon}\right)\ :\ \theta(\cdot)\text{ solves (12)},\theta\left(\tfrac{\tau'_{i+1}}{\varepsilon}\right)\in P_{(i+1)}\right\}.$$

Using Proposition 50 applied to F_{τ_i',σ_i^+} for all $2\leq i\leq n$, we obtain that for every sufficiently small $\varepsilon>0$, if

$$J^-_{(i)}(\varepsilon)\cap J^+_{(i)}(\varepsilon)=\emptyset\quad\forall\,1\leq i<n, \tag{49}$$

then all solutions of (12) starting outside P_ε δ-track S. So, it remains to show that the set of ε-values for which (49) fails is exponentially vanishing. For each $1\leq i<n$, let

$$\mathscr{E}_i=\{\varepsilon>0 : J^-_{(i)}(\varepsilon)\cap J^+_{(i)}(\varepsilon)\neq\emptyset\}.$$

It is clear that a finite union of exponentially vanishing sets is exponentially vanishing, so it is sufficient to show that $\mathscr{E}_i$ is exponentially vanishing for each $1\leq i<n$.

Fix i. By Proposition 51 applied to $F_{\sigma_i^+,\tilde{\sigma}_i}$ and $F_{\tau'_{i+1},\tilde{\sigma}_i}$, we can find $\lambda<0$ such that for all sufficiently small ε, $J^-_{(i)}(\varepsilon)$ and $J^+_{(i)}(\varepsilon)$ are both of length less than $e^{\frac{\lambda}{\varepsilon}}$. Let

$$\mathfrak{s}_i=\begin{cases}1, & F>0\text{ on }\mathbb{S}^1\times(\tau_i^+,\tau_{i+1}^-),\\ -1, & F<0\text{ on }\mathbb{S}^1\times(\tau_i^+,\tau_{i+1}^-).\end{cases}$$

Let $\tilde{\theta}_i^-\in Q_i$ and $\tilde{\theta}_i^+\in P_{(i+1)}$ be such that $\mathfrak{s}_iF(\tilde{\theta}_i^-,\sigma_i^+)>0$ and $\mathfrak{s}_iF(\tilde{\theta}_i^+,\tau'_{i+1})>0$. Let $x_1\colon(0,\infty)\to\mathbb{R}$ be a lift of the function $T\mapsto\Phi(T,\tilde{\theta}_i^-,\sigma_i^+,\tilde{\sigma}_i)$, and let $x_2\colon(0,\infty)\to\mathbb{R}$ be a lift of the function $T\mapsto\Phi(T,\tilde{\theta}_i^+,\tau'_{i+1},\tilde{\sigma}_i)$. Define $x\colon(0,\infty)\to\mathbb{R}$ by $x=\mathfrak{s}_i(x_1-x_2)$, and let $\frac{dx}{d(\varepsilon^{-1})}\colon(0,\infty)\to\mathbb{R}$ denote the derivative of x. For sufficiently small ε, we have

$$\varepsilon\in\bar{\mathscr{E}}_i\implies x\left(\tfrac{1}{\varepsilon}\right)\in\left\{2\pi j+h : j\in\mathbb{N},|h|<2\,e^{\frac{\lambda}{\varepsilon}}\right\}. \tag{50}$$

By Eq. (35) applied to $F_{\sigma_i^+,\tilde{\sigma}_i}$ and $F_{\tau'_{i+1},\tilde{\sigma}_i}$, we have that

$$x\left(\tfrac{1}{\varepsilon}\right)-x(0)\leq\frac{(\tau'_{i+1}-\sigma_i^+)M}{\varepsilon}=:\frac{\mu_+}{\varepsilon} \tag{51}$$

for all $\varepsilon>0$. By Proposition 53 applied to $F_{\sigma_i^+,\tilde{\sigma}_i}$ and $F_{\tau'_{i+1},\tilde{\sigma}_i}$, there exists $\mu_->0$ such that

$$\frac{dx}{d(\varepsilon^{-1})}\geq\mu_-\text{ on the whole of }(0,\infty). \tag{52}$$

Applying elementary analysis to Eqs. (50)–(52) yields that $\bar{\mathscr{E}}_i$ is exponentially vanishing. Specifically, this analysis is as follows:

Given $j\in\mathbb{N}$ and sufficiently small $\varepsilon>0$, if

$$|x\left(\tfrac{1}{\varepsilon}\right)-2\pi j|<2\,e^{\frac{\lambda}{\varepsilon}},$$

then

$$\frac{\mu_+}{\varepsilon}\geq x\left(\tfrac{1}{\varepsilon}\right)-x(0)>2\pi(j-1)-x(0)$$

and so,

$$|x\left(\tfrac{1}{\varepsilon}\right)-2\pi j|<2\,e^{\frac{\lambda(2\pi(j-1)-x(0))}{\mu_+}}\stackrel{\text{def}}{=}\alpha\kappa^j,$$

where $\alpha>0$ and $\kappa\in(0,1)$. So, letting

$$\mathscr{X}_i=\left\{2\pi j+h : j\in\mathbb{N},|h|<\alpha\kappa^j\right\},$$

we have that

$$\varepsilon\in\bar{\mathscr{E}}_i\implies x\left(\tfrac{1}{\varepsilon}\right)\in\mathscr{X}_i$$

for all sufficiently small ε. So, it is sufficient to show that the set $\{\varepsilon>0 : x\left(\tfrac{1}{\varepsilon}\right)\in\mathscr{X}_i\}$ is exponentially vanishing; this is justified as follows: for each $\xi > 0$,

$$\begin{aligned}
&\mathrm{Leb}(\{\zeta \in [\xi,\infty) : x(\zeta) \in \mathscr{X}_i\}) \\
&\quad \le \tfrac{1}{\mu_-}\mathrm{Leb}(\{\zeta \in [x(\xi),\infty) : \zeta \in \mathscr{X}_i\}) \\
&\quad \le \tfrac{1}{\mu_-}\sum_{j=\lfloor x(\xi)\rfloor}^{\infty} 2\alpha\kappa^j \\
&\quad = \frac{2\alpha\kappa^{\lfloor x(\xi)\rfloor}}{\mu_-(1-\kappa)} \\
&\quad < \frac{2\alpha\kappa^{x(\xi)-1}}{\mu_-(1-\kappa)} \\
&\quad \le \frac{2\alpha\kappa^{x(0)-1}}{\mu_-(1-\kappa)}\kappa^{\mu_-\xi}.
\end{aligned}$$

Thus, we have shown that F almost exhibits tracking of (U,y). Combining this with Lemma 46 gives that F is almost exponentially stable with rate Λ_{ad}, by exactly the same reasoning as in the proof of Proposition 51.

I. Transition between neutral stability and exponential stability

Recall from the discussion of the normal form (20) in Sec. III G that the graph of the map $x_2 \mapsto -\sqrt{\mathscr{D}(f_{\gamma,x_2})}$ was a semi-ellipse. Since the quadratic function $F_\gamma^{\mathrm{normal}}$ is only an approximation of the behavior of $(F_\gamma)_{\gamma\in\Gamma}$ around the critical zero, we will need to consider perturbation of ellipses. So, before addressing the specific setting of Sec. III G, let us first give some general notations and a basic fact regarding ellipses and perturbation of ellipses.

For any $\tau_0 \in \mathbb{R}$ and $r_1, r_2 > 0$, define the ellipse

$$\mathscr{S}_{\tau_0,r_1,r_2} = \left\{(x,y) \in \mathbb{R}^2 : \left(\tfrac{x-\tau_0}{r_1}\right)^2 + \left(\tfrac{y}{r_2}\right)^2 = 1\right\},$$

i.e., $\mathscr{S}_{\tau_0,r_1,r_2}$ has center $(\tau_0,0)$ and its principal axes are a horizontal axis of radius r_1 and a vertical axis of radius r_2. We will use the following notation for perturbation of this ellipse: given $\mathbf{h} \in \mathbb{R}^5$ with $r_1 + h_1 > 0$ and $r_2 + h_2 > 0$, define

$$\begin{aligned}
&\tilde{\mathscr{S}}_{\tau_0,r_1,r_2}(\mathbf{h}) \\
&\quad = \left\{(x, y + h_4 + h_5(x-\tau_0)) : \left(\tfrac{x-(\tau_0+h_3)}{r_1+h_1}\right)^2 + \left(\tfrac{y}{r_2+h_2}\right)^2 = 1\right\}.
\end{aligned}$$

Now, for $\Delta \in (0,1)$, define the solid elliptical annulus

$$\mathscr{A}_{\tau_0,r_1,r_2}(\Delta) = \bigcup_{t\in[1-\Delta,1+\Delta]} \mathscr{S}_{\tau_0,tr_1,tr_2}.$$

Also, define $\mathscr{S}^-_{\tau_0,r_1,r_2}$ and $\mathscr{A}^-_{\tau_0,r_1,r_2}(\Delta)$ to be the intersection, respectively, of $\mathscr{S}_{\tau_0,r_1,r_2}$ and $\mathscr{A}_{\tau_0,r_1,r_2}(\Delta)$ with the lower half-plane $\mathbb{R}\times(-\infty,0)$.

We now address the question of how strongly one can perturb the ellipse $\mathscr{S}_{\tau_0,r_1,r_2}$ in the manner described above while remaining within $\mathscr{A}_{\tau_0,r_1,r_2}(\Delta)$.

Lemma 54. *Given $\Delta \in (0,1)$ and $\mathbf{h} \in \mathbb{R}^5$ with $r_1 + h_1 > 0$ and $r_2 + h_2 > 0$, if*

$$\max\left(\tfrac{|h_1|+|h_3|}{r_1}, \tfrac{|h_2|+|h_4|}{r_2} + \tfrac{r_1(1+\frac{1}{\sqrt{2}}\Delta)}{r_2}|h_5|\right) \le \tfrac{1}{\sqrt{2}}\Delta,$$

then $\tilde{\mathscr{S}}_{\tau_0,r_1,r_2}(\mathbf{h}) \subset \mathscr{A}_{\tau_0,r_1,r_2}(\Delta)$.

Proof. First, note that the transformation

$$(\tau_0 + x, y) \mapsto \left(\tfrac{x}{r_1}, \tfrac{y}{r_2}\right)$$

sends $\mathscr{S}_{\tau_0,r_1,r_2}$ and $\mathscr{A}_{\tau_0,r_1,r_2}(\Delta)$ onto $\mathscr{S}_{0,1,1}$ and $\mathscr{A}_{0,1,1}(\Delta)$, respectively, and also sends $\tilde{\mathscr{S}}_{\tau_0,r_1,r_2}(\mathbf{h})$ onto the set $\tilde{\mathscr{S}}_{0,1,1}(\mathbf{h}')$ with

$$\mathbf{h}' = \left(\tfrac{h_1}{r_1}, \tfrac{h_2}{r_2}, \tfrac{h_3}{r_1}, \tfrac{h_4}{r_2}, \tfrac{r_1h_5}{r_2}\right).$$

So, without loss of generality take $\tau_0 = 0$ and $r_1 = r_2 = 1$.

Note that the nearest point on $\partial\mathscr{A}_{0,1,1}(\Delta)$ to any given point on $\mathscr{S}_{0,1,1}$ is of distance Δ away. Now, $\mathscr{S}_{0,1,1}$ is mapped onto $\tilde{\mathscr{S}}_{0,1,1}(\mathbf{h})$ by first applying the transformation

$$(x,y) \mapsto ((1+h_1)x, (1+h_2)y + h_4)$$

and then the transformation

$$(x,y) \mapsto (x, y + h_5x).$$

The first transformation moves points horizontally through a maximum distance of $|h_1| + |h_3|$ and vertically through a maximum distance of $|h_2| + |h_4|$. Since $|h_1| + |h_3| \le \frac{1}{\sqrt{2}}\Delta$, the horizontal coordinate of the resulting points must remain within distance $1 + \frac{1}{\sqrt{2}}\Delta$ of 0. Therefore, the application of the second transformation moves points through a maximum vertical distance of $\left(1 + \frac{1}{\sqrt{2}}\Delta\right)|h_5|$ while leaving the horizontal coordinate the same. Thus, overall, in going from $\mathscr{S}_{0,1,1}$ to $\tilde{\mathscr{S}}_{0,1,1}(\mathbf{h})$, the horizontal and vertical distances moved by each point are a maximum of $\frac{1}{\sqrt{2}}\Delta$, and, therefore, the overall distance is a maximum of Δ. □

Now, given a 2×2 real symmetric matrix A and a vector $\mathbf{x} \in \mathbb{R}^2$, write

$$A[\mathbf{x}] := \mathbf{x}^{\mathsf{T}}A\mathbf{x} = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2.$$

It is easy to verify the following:

(A) If $\det(A) < 0$, then for all $x_2 \in \mathbb{R}\setminus\{0\}$, the quadratic function $x_1 \mapsto A[x_1,x_2]$ takes both positive and negative values.

(B) If $\det(A) > 0$, then $\mathrm{sgn}(a_{11}) = \mathrm{sgn}(a_{22}) \neq 0$, i.e., diag-sgn$(A)$ is well-defined, and furthermore,

$$\mathrm{sgn}(A[x_1,x_2]) = \text{diag-sgn}(A)$$

for all $(x_1,x_2) \in \mathbb{R}^2\setminus\{(0,0)\}$.

Now, for $F \in C^2(\mathbb{S}^1\times[0,1],\mathbb{R})$, if we fix $(\theta,\tau) \in \mathbb{S}^1\times[0,1]$ and $\mathbf{v} \in \mathbb{R}^2$ and define a function $w(\cdot)$ by

$$w(t) = F(\theta + tv_1, \tau + tv_2),$$

then the second derivative of w is given by

$$w''(t) = \mathrm{Hess}_F(\theta + tv_1, \tau + tv_2)[\mathbf{v}]. \tag{53}$$

It is now straightforward to prove Lemma 29.

Proof of Lemma 29. Let us fix vectors $\mathbf{v}, \tilde{\mathbf{v}} \in \mathbb{R}^2$ such that $\mathrm{Hess}_F(\theta_0, \tau_0)[\mathbf{v}] > 0$ and $\mathrm{Hess}_F(\theta_0, \tau_0)[\tilde{\mathbf{v}}] < 0$ and define

$$w(t) = F(\theta + tv_1, \tau + tv_2),$$
$$\tilde{w}(t) = F(\theta + t\tilde{v}_1, \tau + t\tilde{v}_2)$$

for t in a neighborhood of 0. Since $\nabla F(\theta_0, \tau_0) = 0$, we have that $w'(0) = \tilde{w}'(0) = 0$, and, therefore, by Eq. (53), $w(\cdot) > 0$ and $\tilde{w}(\cdot) < 0$ on a punctured neighborhood of 0. It follows, in particular, that $\mathbf{v}$ and $\tilde{\mathbf{v}}$ are linearly independent. Now, for all t in such a punctured neighborhood, the intermediate value theorem gives a point (θ_t, τ_t) on the line segment joining $w(t)$ and $\tilde{w}(t)$ such that $F(\theta_t, \tau_t) = 0$; and since $\mathbf{v}$ and $\tilde{\mathbf{v}}$ are linearly independent, we have that $(\theta_t, \tau_t) \neq (\theta_0, \tau_0)$. □

We now prove Theorem 30.

Standing assumption. Without loss of generality, take $\text{diag-sgn}(D_1) = 1$ and $d_2 > 0$.

For convenience, for all small $\delta > 0$, define

$$\Gamma_\delta = [\gamma_0 - \delta, \gamma_0 + \delta],$$
$$\Theta_\delta = (\theta_0 - \delta, \theta_0 + \delta),$$
$$T_\delta = (\tau_0 - \delta, \tau_0 + \delta),$$
$$S_\delta = \Theta_\delta \times T_\delta \ni (\theta_0, \tau_0),$$
$$C_\delta = \Gamma_\delta \times \bar{S}_\delta \ni (\gamma_0, \theta_0, \tau_0).$$

Note that F_{γ_0} is strictly positive on $(\mathbb{S}^1 \times [0,1]) \setminus \{(\theta_0, \tau_0)\}$. Therefore, by continuity, for all small $\delta > 0$, we can find $r(\delta) \in (0, \delta)$ such that $F_\gamma(\theta, \tau) > 0$ for all

$$(\gamma, \theta, \tau) \in \Gamma_{r(\delta)} \times ((\mathbb{S}^1 \times [0,1]) \setminus S_\delta).$$

Now, take $\delta_1 > 0$ sufficiently small that the following statements hold:

- $\det(\mathrm{Hess}_{F_\gamma}(\theta, \tau)) > 0$ for all $(\gamma, \theta, \tau) \in C_{\delta_1}$ and
- $\mathrm{sgn}(F_\gamma(\theta_0, \tau_0)) = \mathrm{sgn}(\gamma - \gamma_0)$ for all $\gamma \in \Gamma_{\delta_1}$.

Define positive finite constants $\mathfrak{m}_{22}, \mathfrak{M}_{12}$, and $\mathfrak{m}_0$ such that the following statements hold:

- for all $(\gamma, \theta, \tau) \in C_{\delta_1}$ and all $|\mathbf{v}| = 1$,
$$\mathrm{Hess}_{F_\gamma}(\theta, \tau)[\mathbf{v}] \geq \mathfrak{m}_{22};$$
- for all $\gamma \in \Gamma_{\delta_1}$,
$$|\nabla F_\gamma(\theta_0, \tau_0)| \leq \mathfrak{M}_{12}|\gamma - \gamma_0|,$$
$$\mathrm{sgn}(\gamma - \gamma_0) F_\gamma(\theta_0, \tau_0) \geq \mathfrak{m}_0 |\gamma - \gamma_0|.$$

Let $\delta_2 = r(\delta_1)$. We will prove Theorem 30 in the following steps:

- *Step 1.* There does not exist a sequence $(\gamma_n, \theta_n, \tau_n)$ converging to $(\gamma_0, \theta_n, \tau_0)$ such that $F(\gamma_n, \theta_n, \tau_n) = 0$ and $\gamma_n > \gamma_0$ for all n.
  So, we can define $\delta_3 > 0$ sufficiently small that for all $\gamma \in (\gamma_0, \gamma_0 + \delta_3)$, F_γ has no zeros in S_{δ_3}; and hence, for all $\gamma \in (\gamma_0, \gamma_0 + r(\delta_3))$, F_γ has no zeros at all. We then take δ in the statement of Theorem 30 to be $\min(\delta_2, r(\delta_3))$.
- *Step 2.* For all $\gamma \in (\gamma_0 - \delta_2, \gamma_0)$, F_γ is generic with a unique complete curve (U_γ, y_γ) of the stable slow manifold, and U_γ is non-empty and connected. Write
$$Y_\gamma(\tau) := \tfrac{\partial F_\gamma}{\partial \theta}(y_\gamma(\tau), \tau),$$
so that
$$\Lambda(\gamma) = \int_{U_\gamma} Y_\gamma(\tau)\, d\tau.$$
- *Step 3.* We will propose an approximation $(\tilde{U}_\gamma, \tilde{Y}_\gamma)$ of (U_γ, Y_γ), based on the normal form (20); this gives
$$\tilde{\Lambda}(\gamma) := \int_{\tilde{U}_\gamma} \tilde{Y}_\gamma(\tau)\, d\tau = \frac{\pi d_{\theta\theta} d_2(\gamma - \gamma_0)}{\sqrt{\det(D_1)}}.$$
- *Step 4.* For any $\eta > 0$, taking $\gamma > \gamma_0$ sufficiently close to γ_0 guarantees that
$$|\Lambda(\gamma) - \tilde{\Lambda}(\gamma)| \leq \eta |\gamma - \gamma_0|.$$

1. Step 1

Suppose for a contradiction that such a sequence $(\gamma_n, \theta_n, \tau_n)$ exists. For all sufficiently large n, we have that $\gamma_n \in \Gamma_{\delta_1}$, and so $(\theta_n, \tau_n) \neq (\theta_0, \tau_n)$. So, let $t_n > 0$ and $\mathbf{v}^{(n)}$ with $|\mathbf{v}^{(n)}| = 1$ be such that

$$(\theta_n, \tau_n) = (\theta_0 + t_n v_1^{(n)}, \tau_0 + t_n v_2^{(n)}).$$

By Taylor's theorem, there exists $\tilde{t}_n \in (0, t_n)$ such that, letting

$$(\tilde{\theta}_n, \tilde{\tau}_n) = (\theta_0 + \tilde{t}_n v_1^{(n)}, \tau_0 + \tilde{t}_n v_2^{(n)}),$$

we have

$$0 = F_{\gamma_n}(\theta_0, \tau_0) + (\nabla F_{\gamma_n}(\theta_0, \tau_0) \cdot \mathbf{v}^{(n)}) t_n + \tfrac{1}{2} \mathrm{Hess}_{F_{\gamma_n}}(\tilde{\theta}_n, \tilde{\tau}_n)[\mathbf{v}^{(n)}] t_n^2$$

and, therefore,

$$(\nabla F_{\gamma_n}(\theta_0, \tau_0) \cdot \mathbf{v}^{(n)})^2 \geq 2 F_{\gamma_n}(\theta_0, \tau_0) \mathrm{Hess}_{F_{\gamma_n}}(\tilde{\theta}_n, \tilde{\tau}_n)[\mathbf{v}^{(n)}].$$

For sufficiently large n, we have $(\gamma_n, \tilde{\theta}_n, \tilde{\tau}_n) \in C_{\delta_1}$ and so it follows that

$$\mathfrak{M}_{12}^2 (\gamma_n - \gamma_0)^2 \geq 2 \mathfrak{m}_0 \mathfrak{m}_{22} (\gamma_n - \gamma_0)$$

and, therefore,

$$\mathfrak{M}_{12}^2 (\gamma_n - \gamma_0) \geq 2 \mathfrak{m}_0 \mathfrak{m}_{22}.$$

However, this contradicts the fact that $\gamma_n \to \gamma_0$ as $n \to \infty$.

2. Step 2

Fix $\gamma \in (\gamma_0 - \delta_2, \gamma_0)$. Recall that F_γ is strictly positive outside S_{δ_1} and so all zeros of F_γ lie inside S_{δ_1}. For each $\tau \in T_{\delta_1}$, we have that $\mathrm{Hess}_{F_\gamma}(\cdot)[1, 0]$ is strictly positive on the whole of $\Theta_{\delta_1} \times \{\tau\}$ and so $\Theta_{\delta_1} \times \{\tau\}$ contains at most two zeros, and if $\Theta_{\delta_1} \times \{\tau\}$ contains two zeros, then one is hyperbolic stable [which we denote $(y_\gamma(\tau), \tau)$], and the other is hyperbolic unstable [which we denote $(z_\gamma(\tau), \tau)$]. Let

$$U_\gamma := \{\tau \in T_{\delta_1} : \Theta_{\delta_1} \times \{\tau\} \text{ has two zeros}\}.$$

By Remark 9(A), U_γ is open and $y_\gamma(\cdot)$ and $z_\gamma(\cdot)$ are continuous on U_γ. Now, $F_\gamma(\tau_0, \theta_0) < 0$, and so by the intermediate value theorem,

$\Theta_{\delta_1} \times \{\tau_0\}$ has at least (and hence exactly) two zeros. In other words, $\tau_0 \in U_\gamma$. So, let $(\tau_\gamma^-, \tau_\gamma^+)$ be the connected component of U_γ containing τ_0. Since the set of zeros of F is closed, it is clear that $\Theta_{\delta_1} \times \{\tau_\gamma^-\}$ and $\Theta_{\delta_1} \times \{\tau_\gamma^+\}$ each contain a zero of F, which we denote $(\theta_\gamma^-, \tau_\gamma^-)$ and $(\theta_\gamma^+, \tau_\gamma^+)$, and it is clear that both $y_\gamma(\tau)$ and $z_\gamma(\tau)$ converge to $\theta_\gamma^\pm$ as $\tau \to \tau_\gamma^\pm$. Hence, in particular, by Remark 9(A), $(\theta_\gamma^-, \tau_\gamma^-)$ and $(\theta_\gamma^+, \tau_\gamma^+)$ are non-hyperbolic zeros.

We next show that $(\theta_\gamma^-, \tau_\gamma^-)$ and $(\theta_\gamma^+, \tau_\gamma^+)$ are non-degenerate. By definition of δ_1, since $(\gamma, \theta_\gamma^\pm, \tau_\gamma^\pm) \in C_{\delta_1}$, we have $\frac{\partial^2 F_\gamma}{\partial\theta^2}(\theta_\gamma^\pm, \tau_\gamma^\pm) > 0$. Now, take $\mathbf{v}^\pm \in \mathbb{R}^2$ such that

$$(\theta_\gamma^\pm, \tau_\gamma^\pm) = (\theta_0 + v_1^\pm, \tau_0 + v_2^\pm).$$

Since $F_\gamma(\theta_0, \tau_0) < 0$ and $\mathrm{Hess}_{F_\gamma}(\cdot)[\mathbf{v}^\pm]$ is strictly positive throughout the line segment

$$\{(\theta_0 + tv_1^\pm, \tau_0 + tv_2^\pm) : t \in [0, 1]\},$$

we have that

$$\nabla F_\gamma(\theta_\gamma^\pm, \tau_\gamma^\pm) \cdot \mathbf{v}^\pm > 0.$$

Since $\frac{\partial F}{\partial\theta}(\theta_\gamma^\pm, \tau_\gamma^\pm) = 0$, it follows that $\frac{\partial F}{\partial\tau}(\theta_\gamma^\pm, \tau_\gamma^\pm) \neq 0$.

Finally, we show that there are no zeros of F_γ other than those which we have already mentioned, namely, those in the set

$$\{(y_\gamma(\tau), \tau), (z_\gamma(\tau), \tau)\}_{\tau \in (\tau_\gamma^-, \tau_\gamma^+)} \cup \{(\theta_\gamma^\pm, \tau_\gamma^\pm)\}.$$

By definition, all zeros in $\Theta_{\delta_1} \times (\tau_\gamma^-, \tau_\gamma^+)$ belong to this set. So, it remains to show that given any

$$(\theta, \tau) \in \Theta_{\delta_1} \times (T_\delta \setminus (\tau_\gamma^-, \tau_\gamma^+)),$$

we have $F_\gamma(\theta, \tau) > 0$. Assume that $\tau \in (\tau_0 - \delta_1, \tau_\gamma^-)$; the case that $\tau \in (\tau_\gamma^+, \tau_0 + \delta_1)$ proceeds in exactly the same way. Take $t' > 1$ and $\mathbf{v} \in \mathbb{R}^2$ such that

$$\tau_\gamma^- = \tau_0 + v_2,$$
$$(\theta, \tau) = (\theta_0 + t'v_1, \tau_0 + t'v_2).$$

Once again, we have that $F_\gamma(\theta_0, \tau_0) < 0$ and $\mathrm{Hess}_{F_\gamma}(\cdot)[\mathbf{v}]$ is strictly positive throughout the line segment

$$\{\mathbf{s}(t) := (\theta_0 + tv_1, \tau_0 + tv_2) : t \in [0, t']\}.$$

Therefore, splitting into the cases that

(i) $\theta_0 + v_1 = \theta_\gamma^-$,
(ii) $\theta_0 + v_1 \in (\theta_\gamma^-, \theta_0 + \delta_1)$,
(iii) $\theta_0 + v_1 \in (\theta_0 - \delta_1, \theta_\gamma^-)$,

we have the following: In case (i), $F_\gamma(\mathbf{s}(1)) = 0$ and so $F_\gamma(\theta, \tau) = F_\gamma(\mathbf{s}(t')) > 0$. In case (ii), by the intermediate value theorem, there exists $t'' \in (0, 1)$ such that, letting $\tau'' = \tau_0 + t''v_2$, we have

$$\theta_0 + t''v_1 = z_\gamma(\tau''),$$

so $F_\gamma(\mathbf{s}(t'')) = 0$ and hence, again, $F_\gamma(\theta, \tau) > 0$. Case (iii) is the same, with y_γ in place of z_γ.

3. Step 3

For convenience, let $\rho_1 = \sqrt{\frac{2d_{\theta\theta}d_2}{\det(D_1)}}$ and $\rho_2 = \sqrt{2d_{\theta\theta}d_2}$. For each $\gamma < \gamma_0$, define the set $\tilde{U}_\gamma$ and the negative-valued function $\tilde{Y}_\gamma(\cdot)$ on $\tilde{U}_\gamma$ such that

$$\operatorname{graph} \tilde{Y}_\gamma = \mathscr{S}^-_{\tau_0, \rho_1\sqrt{\gamma_0-\gamma}, \rho_2\sqrt{\gamma_0-\gamma}}.$$

For each $\tau \in \tilde{U}_\gamma$, we can write $\tilde{Y}_\gamma(\tau)$ more explicitly as

$$\tilde{Y}_\gamma(\tau)^2 = -\det(D_1)(\tau - \tau_0)^2 + 2d_{\theta\theta}d_2(\gamma_0 - \gamma).$$

4. Step 4

Fix $\eta > 0$, and let $\Delta \in (0, 1)$ be sufficiently small that the area of the elliptical semi-annulus $\mathscr{A}^-_{0,\rho_1,\rho_2}(\Delta)$ is at most η. For each $\gamma \in (\gamma_0 - \delta_2, \gamma_0)$, we have that $Y_\gamma < 0$ on U_γ and Y_γ tends to 0 at the boundary points of U_γ. Therefore, to show the desired result, it is sufficient to show that for all $\gamma < \gamma_0$ sufficiently close to γ we have

$$\operatorname{graph} Y_\gamma \subset \mathscr{A}_{\tau_0, \rho_1\sqrt{\gamma_0-\gamma}, \rho_2\sqrt{\gamma_0-\gamma}}(\Delta).$$

For each $\gamma \in (\gamma_0 - \delta_2, \gamma_0)$, let $\delta_{(\gamma)} > 0$ be the smallest value for which all zeros of F_γ are contained in $\bar{S}_{\delta_{(\gamma)}}$. Note that $\delta_{(\gamma)} \to 0$ as $\gamma \nearrow \gamma_0$: for any small $\tilde{\eta} > 0$, taking $\gamma \in (\gamma_0 - r(\tilde{\eta}), \gamma_0)$ gives $\delta_{(\gamma)} < \tilde{\eta}$.

Now, for each $\gamma \in (\gamma_0 - \delta_2, \gamma_0)$, for each $\tau \in U_\gamma$, we can apply Taylor's theorem to the functions $F_\gamma(\theta_0, \cdot)$, $F_\gamma(\cdot, \tau)$, and $\frac{\partial F_\gamma}{\partial\theta}(\theta_0, \cdot)$, respectively, to obtain expressions

$$F_\gamma(\theta_0, \tau) = F_\gamma(\theta_0, \tau_0) + \tfrac{\partial F_\gamma}{\partial\tau}(\theta_0, \tau_0)(\tau - \tau_0) + \tfrac{1}{2}\tilde{d}^{\gamma,\tau}_{\tau\tau}(\tau - \tau_0)^2,$$
$$F_\gamma(\theta_0, \tau) = \tfrac{\partial F_\gamma}{\partial\theta}(\theta_0, \tau)\, d(\theta_0, y_\gamma(\tau)) - \tfrac{1}{2}\tilde{d}^{\gamma,\tau}_{\theta\theta}\, d(\theta_0, y_\gamma(\tau))^2,$$
$$\tfrac{\partial F_\gamma}{\partial\theta}(\theta_0, \tau) = \tfrac{\partial F_\gamma}{\partial\theta}(\theta_0, \tau_0) + \tilde{d}^{\gamma,\tau}_{\theta\tau}(\tau - \tau_0),$$

where $\tilde{d}^{\gamma,\tau}_{\tau\tau}$, $\tilde{d}^{\gamma,\tau}_{\theta\theta}$, and $\tilde{d}^{\gamma,\tau}_{\theta\tau}$ are, respectively, the values of $\frac{\partial^2 F_\gamma}{\partial\tau^2}$, $\frac{\partial^2 F_\gamma}{\partial\theta^2}$, and $\frac{\partial^2 F_\gamma}{\partial\theta\partial\tau}$ at certain points in $S_{\delta_{(\gamma)}}$. Since $\delta_{(\gamma)} \to 0$ as $\gamma \nearrow \gamma_0$, letting

$$\tilde{D}_1^{\gamma,\tau} := \begin{pmatrix} \tilde{d}^{\gamma,\tau}_{\theta\theta} & \tilde{d}^{\gamma,\tau}_{\theta\tau} \\ \tilde{d}^{\gamma,\tau}_{\theta\tau} & \tilde{d}^{\gamma,\tau}_{\tau\tau} \end{pmatrix},$$

we have that $\tilde{D}_1^{\gamma,\tau}$ converges to D_1 uniformly across $\tau \in U_\gamma$ as $\gamma \nearrow \gamma_0$. Also, recall that $|\nabla F_\gamma(\theta_0, \tau_0)|$ is $O(\gamma_0 - \gamma)$ as $\gamma \nearrow \gamma_0$. Therefore, in the above three equations, if we substitute the expressions for $F_\gamma(\theta_0, \tau)$ and $\frac{\partial F_\gamma}{\partial\theta}(\theta_0, \tau)$ given by the first and third equation into the second equation and then rearrange the second equation to make $y_\gamma(\tau)$ the subject, we obtain an expression of the form

$$y_\gamma(\tau) = \theta_0 - \frac{\upsilon_{\gamma,\tau} + \sqrt{\tilde{Y}_\gamma(\tau)^2 + E_{\gamma,\tau}}}{\tilde{d}^{\gamma,\tau}_{\theta\theta}}, \tag{54}$$

where $\upsilon_{\gamma,\tau} = \frac{\partial F_\gamma}{\partial\theta}(\theta_0, \tau)$ is as given by the third equation, and $E_{\gamma,\tau}$ takes the form

$$E_{\gamma,\tau} = a_{\gamma,\tau}(\tau - \tau_0)^2 + b_{\gamma,\tau}(\tau - \tau_0) + c_{\gamma,\tau},$$

where

- $a_{\gamma,\tau} \to 0$,
- $b_{\gamma,\tau}$ is $O(\gamma_0 - \gamma)$, and
- $c_{\gamma,\tau}$ is $o(\gamma_0 - \gamma)$,

uniformly across $\tau \in U_\gamma$ as $\gamma \nearrow \gamma_0$. Now, by applying Taylor's theorem to $\frac{\partial F_\gamma}{\partial \theta}(\cdot, \tau)$, using Eq. (54), we obtain an expression

$$Y_\gamma(\tau) = \upsilon_{\gamma,\tau} - \tilde{\tilde{d}}_{\theta\theta}^{\gamma,\tau} \left(\frac{\upsilon_{\gamma,\tau} + \sqrt{\tilde{Y}_\gamma(\tau)^2 + E_{\gamma,\tau}}}{\tilde{d}_{\theta\theta}^{\gamma,\tau}} \right),$$

where $\tilde{\tilde{d}}_{\theta\theta}^{\gamma,\tau}$ is the value of $\frac{\partial^2 F_\gamma}{\partial \theta^2}$ at some point in $S_{\delta(\gamma)}$. Again, $\tilde{\tilde{d}}_{\theta\theta}^{\gamma,\tau} \to d_{\theta\theta}$ uniformly across $\tau \in U_\gamma$ as $\gamma \nearrow \gamma_0$. Therefore, one obtains that, after suitable modification of the coefficients $a_{\gamma,\tau}, b_{\gamma,\tau}, c_{\gamma,\tau}$ (while keeping the same convergence properties as above), we have

$$Y_\gamma(\tau) = -\sqrt{\tilde{Y}_\gamma(\tau)^2 + E_{\gamma,\tau}} + h_4^{\gamma,\tau} + h_5^{\gamma,\tau}(\tau - \tau_0),$$

where

$$h_4^{\gamma,\tau} = \frac{\partial F_\gamma}{\partial \theta}(\theta_0, \tau_0) \left(1 - \frac{\tilde{\tilde{d}}_{\theta\theta}^{\gamma,\tau}}{\tilde{d}_{\theta\theta}^{\gamma,\tau}} \right),$$

$$h_5^{\gamma,\tau} = \tilde{d}_{\theta\tau}^{\gamma,\tau} \left(1 - \frac{\tilde{\tilde{d}}_{\theta\theta}^{\gamma,\tau}}{\tilde{d}_{\theta\theta}^{\gamma,\tau}} \right).$$

Now, for convenience, let us write $a := -\det(D_1)$ and $c_\gamma := 2 d_{\theta\theta} d_2(\gamma_0 - \gamma)$, so that

$$\tilde{Y}_\gamma(\tau)^2 = a(\tau - \tau_0)^2 + c_\gamma.$$

The point $(\tau, Y_\gamma(\tau))$ lies on the curve

$$\tilde{\mathscr{S}}_{\tau_0, \rho_1\sqrt{\gamma_0-\gamma}, \rho_2\sqrt{\gamma_0-\gamma}}(\mathbf{h}^{\gamma,\tau}),$$

with $h_4^{\gamma,\tau}$ and $h_5^{\gamma,\tau}$ as above and

$$h_1^{\gamma,\tau} = \sqrt{b_{\gamma,\tau}^2 - 4(a + a_{\gamma,\tau})(c_\gamma + c_{\gamma,\tau})} - \sqrt{-4ac_\gamma},$$

$$h_2^{\gamma,\tau} = \sqrt{c_\gamma + c_{\gamma,\tau}} - \sqrt{c_\gamma},$$

$$h_3^{\gamma,\tau} = -\frac{b_{\gamma,\tau}}{2(a + a_{\gamma,\tau})}.$$

The general inequalities $|\sqrt{A} - \sqrt{B}| \leq \sqrt{|A - B|}$ and $\sqrt{A + B} \leq \sqrt{|A|} + \sqrt{|B|}$ (understood as valid for all $A, B \in \mathbb{R}$ for which they are well-defined) give

$$|h_1^{\gamma,\tau}| \leq |b_{\gamma,\tau}| + 2\sqrt{|ac_{\gamma,\tau} + a_{\gamma,\tau}(c_\gamma + c_{\gamma,\tau})|},$$

$$|h_2^{\gamma,\tau}| \leq \sqrt{|c_{\gamma,\tau}|}.$$

So,

- $h_1^{\gamma,\tau}$ and $h_2^{\gamma,\tau}$ are $o(\sqrt{\gamma_0 - \gamma})$,
- $h_3^{\gamma,\tau}$ and $h_4^{\gamma,\tau}$ are, respectively, $O(\gamma_0 - \gamma)$ and $o(\gamma_0 - \gamma)$, and hence are both $o(\sqrt{\gamma_0 - \gamma})$, and
- $h_5^{\gamma,\tau} \to 0$,

uniformly across $\tau \in U_\gamma$ as $\gamma \nearrow \gamma_0$. Hence, in particular, taking γ sufficiently close to γ_0 will mean that $\mathbf{h}^{\gamma,\tau}$ fulfills the condition in Lemma 54 (with $r_i = \rho_i\sqrt{\gamma_0 - \gamma}$) for all $\tau \in U_\gamma$, and hence

$$\text{graph } Y_\gamma \subset \mathscr{A}_{\tau_0, \rho_1\sqrt{\gamma_0-\gamma}, \rho_2\sqrt{\gamma_0-\gamma}}(\Delta).$$

This completes the proof.

VI. CONCLUSION

The modern theory of dynamical systems was introduced by Henri Poincaré and Aleksandr Lyapunov to provide a rigorous mathematical basis for defining and investigating qualitative dynamical properties of real-world physical systems—properties such as synchronization, stability, and neutral stability. Using a numerically simulated sample realization of a Brownian bridge as a conceptual representation of inherently finite-time processes, we have illustrated in Fig. 2 how a finite-time system can exhibit precisely such properties in a way that cannot be modeled so as to make the framework of Poincaré and Lyapunov applicable. This is because this framework is defined in terms of the infinite-time behavior of dynamical systems.

Therefore, we have introduced an alternative, finite-time framework for stability analysis by essentially "replacing $t \to \infty$ with $\varepsilon \to 0$" in the traditional formalisms of qualitative stability analysis, where ε represents the timescale separation between a system's "internal timescales" and the slower timescale of a finite-time external forcing process. This non-traditional framework has enabled us to provide a rigorous mathematical statement (Theorem 24) and proof of the stabilization phenomenon of Refs. 1 and 2 exemplified in Fig. 2.

We have also explored how our new framework of stability analysis compares with the traditional framework through the example of low-frequency-periodically forced systems, where both frameworks can be applied. In particular, we saw in Sec. IV E how analysis based on Theorem 24 can more readily yield the basic description of dynamics than seeking to analyze the system within the traditional apparatus of stability analysis (such as asymptotic Lyapunov exponents).

This paper has worked entirely with one-dimensional dynamics but we hope that the work here will also prompt further research into non-traditional qualitative stability analysis of higher-dimensional finite-time systems. Whereas the work of Poincaré was originally motivated by celestial mechanics, we anticipate that finite-time stability theory will increasingly play a fundamental and necessary role in the understanding of more complex "terrestrial" systems such as biological processes and climate systems.

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AUTHOR DECLARATIONS

Conflicts of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Codes for the numerics carried out in this paper (as described in Appendix B) are available from the corresponding author upon reasonable request and are openly available in Lancaster Publications and Research system Pure at https://doi.org/10.17635/lancaster/researchdata/292, Ref. 51.

APPENDIX A: LIMITING BEHAVIOR OF $\frac{\partial\Phi}{\partial(\varepsilon^{-1})}$

Proposition 26 can be expressed as a statement about the limiting behavior of $\frac{\partial\Phi}{\partial\theta}\left(\frac{1}{\varepsilon},\theta_0,0,1\right)$ as $\varepsilon\to 0$. We now give an analogous statement about the limiting behavior of $\frac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\frac{1}{\varepsilon},\theta_0,0,1\right)$ as $\varepsilon\to 0$. Let us first make a couple of general remarks regarding $\frac{\partial\Phi}{\partial(\varepsilon^{-1})}$:

(A) Given $t_1,\dots,t_n\in\left[0,\frac{1}{\varepsilon}\right]$, using Eq. (34), we have

$$\begin{aligned}
&\frac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon},\theta(t_1),\varepsilon t_1,\varepsilon t_n\right)\\
&= \varepsilon\int_{t_1}^{t_n} e^{\bar\lambda^{\varepsilon,F}_{u,t_n}(\theta_0)}F(\theta(u),\varepsilon u)\,du\\
&= \varepsilon\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}} e^{\bar\lambda^{\varepsilon,F}_{u,t_n}(\theta_0)}F(\theta(u),\varepsilon u)\,du\\
&= \varepsilon\sum_{i=1}^{n-1} e^{\bar\lambda^{\varepsilon,F}_{t_{i+1},t_n}(\theta_0)}\int_{t_i}^{t_{i+1}} e^{\bar\lambda^{\varepsilon,F}_{u,t_{i+1}}(\theta_0)}F(\theta(u),\varepsilon u)\,du\\
&= \sum_{i=1}^{n-1} e^{\bar\lambda^{\varepsilon,F}_{t_{i+1},t_n}(\theta_0)}\frac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon},\theta(t_i),\varepsilon t_i,\varepsilon t_{i+1}\right). \qquad \text{(A1)}
\end{aligned}$$

(B) Suppose $F(\cdot,\tau)=:f(\cdot)$ is independent of τ [meaning that (12) is just a time-restricted autonomous dynamical system]. Then, $\frac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\frac{1}{\varepsilon},\theta(s),\varepsilon s,\varepsilon t\right)$ is, by definition, the derivative of $T\mapsto\theta(s+\varepsilon(t-s)T)$ at $T=\frac{1}{\varepsilon}$, which is $\varepsilon(t-s)\dot\theta(t)$. In other words, we have

$$\frac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon},\theta(s),\varepsilon s,\varepsilon t\right)=\varepsilon(t-s)f(\theta(t)). \qquad \text{(A2)}$$

We use the notations present in Proposition 26, as well as the following:

$$m(\tau)=\max_{\theta\in\mathbb{S}^1}|F(\theta,\tau)|,$$

$$l(\tau)=\min_{\theta\in\mathbb{S}^1}|F(\theta,\tau)|.$$

Proposition A.55. *Suppose F has no zeros. Then,*

$$\liminf_{\varepsilon\to 0}\min_{\theta_0\in\mathbb{S}^1}\operatorname{sgn}(F)\frac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon},\theta_0,0,1\right)\ge\int_0^1 l(\tau)\,e^{-\int_\tau^1 r(\sigma)\,d\sigma}\,d\tau$$

and

$$\limsup_{\varepsilon\to 0}\max_{\theta_0\in\mathbb{S}^1}\operatorname{sgn}(F)\frac{\partial\Phi}{\partial(\varepsilon^{-1})}\left(\tfrac{1}{\varepsilon},\theta_0,0,1\right)\le\int_0^1 m(\tau)\,e^{\int_\tau^1 r(\sigma)\,d\sigma}\,d\tau.$$

Proof. Given $\varepsilon>0$, define an integer $N\ge 1$ and times

$$\tfrac{1}{\varepsilon}=\tilde t_1>\cdots>\tilde t_{N+1}=0,$$

such that

- $\tilde t_{i+1}=\tilde t_i-k(\varepsilon\tilde t_i)$ for each $1\le i<N$ and
- $0\in[\tilde t_N-k(\varepsilon\tilde t_N),\tilde t_N)$.

For each $1\le i\le N$, for any continuous function $g\colon[0,1]\to\mathbb{R}$, let

$$g^{\tilde\vee i}=\max_{\tau\in[\varepsilon\tilde t_{i+1},\varepsilon\tilde t_i]}g(\tau),$$

and let $\tilde r_i$ be as in Eq. (37). Similarly to the proof of Proposition 26, we have that

$$\max_{1\le i<N}(\tilde r_i-r(\varepsilon\tilde t_i))\to 0$$

and hence,

$$\max_{1\le i<N}\left|\left(\sum_{j=1}^{i-1}\varepsilon k(\varepsilon\tilde t_j)\tilde r_j\right)-\int_{\varepsilon\tilde t_i}^1 r(\sigma)\,d\sigma\right|\to 0,$$

as $\varepsilon\to 0$. Hence,

$$\sum_{i=1}^{N-1}\varepsilon k(\varepsilon\tilde t_i)l(\varepsilon\tilde t_i)\,e^{-\sum_{j=1}^{i-1}\varepsilon k(\varepsilon\tilde t_j)\tilde r_j}\to\int_0^1 l(\tau)\,e^{-\int_\tau^1 r(\sigma)\,d\sigma}\,d\tau \qquad \text{(A3)}$$

and

$$\sum_{i=1}^{N-1}\varepsilon k(\varepsilon\tilde t_i)m(\varepsilon\tilde t_i)\,e^{\sum_{j=1}^{i-1}\varepsilon k(\varepsilon\tilde t_j)\tilde r_j}\to\int_0^1 m(\tau)\,e^{\int_\tau^1 r(\sigma)\,d\sigma}\,d\tau \qquad \text{(A4)}$$

as $\varepsilon\to 0$. Now, letting $\theta(\cdot)$ be a solution of (12) with $\theta(0)=\theta_0$, as in the proof of Proposition 26, we have

$$|\bar\lambda^{\varepsilon,F}_{\tilde t_{i+1},\tilde t_i}(\theta_0)|\le\varepsilon k(\varepsilon\tilde t_i)\tilde r_i,$$

and so the left-hand sides of (A3) and (A4) are, respectively, a lower and an upper bound of the expression

$$d(\theta_0,\varepsilon):=\sum_{i=1}^{N-1}\varepsilon k(\varepsilon\tilde t_i)|F(\theta(\tilde t_i),\varepsilon\tilde t_i)|\,e^{\bar\lambda^{\varepsilon,F}_{\tilde t_i,\frac{1}{\varepsilon}}(\theta_0)}.$$

Therefore, to prove the desired result, we will show that there are constants $\varepsilon_0>0$ and $E\ge 0$ (dependent only on F) such that for all

$\varepsilon \in (0, \varepsilon_0)$ and $\theta_0 \in \mathbb{S}^1$,

$$D(\theta_0, \varepsilon) := \left| \operatorname{sgn}(F) \tfrac{\partial \Phi}{\partial(\varepsilon^{-1})} \left(\tfrac{1}{\varepsilon}, \theta_0, 0, 1\right) - d(\theta_0, \varepsilon) \right| \leq \varepsilon E. \qquad \text{(A5)}$$

Given $\varepsilon > 0$ and $\theta_0 \in \mathbb{S}^1$, let $\theta(\cdot)$ be the solution of (12) with $\theta(0) = \theta_0$, and let

$$d_i = \tfrac{\partial \Phi}{\partial(\varepsilon^{-1})} \left(\tfrac{1}{\varepsilon}, \theta(\tilde{t}_{i+1}), \varepsilon\tilde{t}_{i+1}, \varepsilon\tilde{t}_i\right)$$

for each $1 \leq i \leq N$. By Eq. (A1),

$$\tfrac{\partial \Phi}{\partial(\varepsilon^{-1})} \left(\tfrac{1}{\varepsilon}, \theta_0, 0, 1\right) = \sum_{i=1}^{N} d_i \, e^{\bar{\lambda}^{\varepsilon,F}_{\tilde{t}_i, \frac{1}{\varepsilon}}(\theta_0)},$$

and it is clear from Eq. (34) that $\operatorname{sgn}(F) d_i \geq 0$, and so we have

$$\begin{aligned} D(\theta_0, \varepsilon) &\leq d_N \, e^{\bar{\lambda}^{\varepsilon,F}_{\tilde{t}_N, \frac{1}{\varepsilon}}(\theta_0)} + \sum_{i=1}^{N-1} |d_i - \varepsilon k(\varepsilon\tilde{t}_i) F(\theta(\tilde{t}_i), \varepsilon\tilde{t}_i)| e^{\bar{\lambda}^{\varepsilon,F}_{\tilde{t}_i, \frac{1}{\varepsilon}}(\theta_0)} \\ &\leq R. \left(d_N + \sum_{i=1}^{N-1} |d_i - \varepsilon k(\varepsilon\tilde{t}_i) F(\theta(\tilde{t}_i), \varepsilon\tilde{t}_i)| \right), \end{aligned}$$

where R is as in the proof of Theorem 24(A). Letting $K = \max_{\tau \in [0,1]} k(\tau)$, Eq. (34) implies

$$d_N \leq \varepsilon K M \, e^{K M_1},$$

and so it remains to find an ε-independent bound on $\frac{1}{\varepsilon} \sum_{i=1}^{N-1} |d_i - \varepsilon k(\varepsilon\tilde{t}_i) F(\theta(\tilde{t}_i), \varepsilon\tilde{t}_i)|$. For each $1 \leq i < N$, let $\psi_i(\cdot)$ be the solution of

$$\dot{\psi}_i(t) = F(\psi_i(t), \varepsilon\tilde{t}_i)$$

for which $\psi_i(\tilde{t}_i) = \theta(\tilde{t}_i)$. Writing F_i for the τ-independent function $F_i(\theta, \tau) = F(\theta, \varepsilon\tilde{t}_i)$, Eqs. (34) and (A2) applied to F_i give

$$\varepsilon \int_{\tilde{t}_{i+1}}^{\tilde{t}_i} e^{\bar{\lambda}^{\varepsilon,F_i}_{u,\tilde{t}_i}(\psi_i(0))} F(\psi_i(u), \varepsilon\tilde{t}_i) \, du = \varepsilon k(\varepsilon\tilde{t}_i) F(\theta(\tilde{t}_i), \varepsilon\tilde{t}_i).$$

Hence, applying Eq. (34) also to d_i, we have

$$\begin{aligned} &\tfrac{1}{\varepsilon} |d_i - \varepsilon k(\varepsilon\tilde{t}_i) F(\theta(\tilde{t}_i), \varepsilon\tilde{t}_i)| \\ &\quad = \left| \int_{\tilde{t}_{i+1}}^{\tilde{t}_i} e^{\bar{\lambda}^{\varepsilon,F}_{u,\tilde{t}_i}(\theta_0)} F(\theta(u), \varepsilon u) - e^{\bar{\lambda}^{\varepsilon,F_i}_{u,\tilde{t}_i}(\psi_i(0))} F(\psi_i(u), \varepsilon\tilde{t}_i) \, du \right|. \end{aligned}$$

By the same reasoning as in the proof of Proposition 26, for each $u \in [\tilde{t}_{i+1}, \tilde{t}_i]$, we have

$$\begin{aligned} &|\bar{\lambda}^{\varepsilon,F}_{u,\tilde{t}_i}(\theta_0) - \bar{\lambda}^{\varepsilon,F_i}_{u,\tilde{t}_i}(\psi_i(0))| \\ &\quad \leq \varepsilon \int_u^{\tilde{t}_i} m_{11}^{\tilde{\triangledown} i} m_2^{\tilde{\triangledown} i} \mathfrak{e}_2(m_1^{\tilde{\triangledown} i}(\tilde{t}_i - t))(\tilde{t}_i - t)^2 + m_{12}^{\tilde{\triangledown} i}(\tilde{t}_i - t) \, dt \\ &\quad \leq \varepsilon K^2 (M_{11} M_2 \mathfrak{e}_2(M_1 K) K + M_{12}) =: \varepsilon \varepsilon_0^{-1}. \end{aligned}$$

So, if $\varepsilon < \varepsilon_0$, then

$$\left| e^{\bar{\lambda}^{\varepsilon,F}_{u,\tilde{t}_i}(\theta_0)} - e^{\bar{\lambda}^{\varepsilon,F_i}_{u,\tilde{t}_i}(\psi_i(0))} \right| \leq \varepsilon \varepsilon_0^{-1} \, e^{\bar{\lambda}^{\varepsilon,F_i}_{u,\tilde{t}_i}(\psi_i(0))+1} \leq \varepsilon \varepsilon_0^{-1} \, e^{K M_1 + 1}.$$

We also have, again by the same reasoning as in Proposition 26, that for each $u \in [\tilde{t}_{i+1}, \tilde{t}_i]$,

$$\begin{aligned} &|F(\theta(u), \varepsilon u) - F(\psi_i(u), \varepsilon\tilde{t}_i)| \\ &\quad \leq m_1^{\tilde{\triangledown} i} |\hat{\theta}(u) - \hat{\psi}_i(u)| + m_2^{\tilde{\triangledown} i} \varepsilon(\tilde{t}_i - u) \\ &\quad \leq \varepsilon (m_1^{\tilde{\triangledown} i} m_2^{\tilde{\triangledown} i} \mathfrak{e}_2(m_1^{\tilde{\triangledown} i}(\tilde{t}_i - u))(\tilde{t}_i - u)^2 + m_2^{\tilde{\triangledown} i}(\tilde{t}_i - u)) \\ &\quad \leq \varepsilon K^2 M_2 (M_1 \mathfrak{e}_2(M_1 K) K + 1) =: \varepsilon E_1. \end{aligned}$$

Hence, if $\varepsilon < \varepsilon_0$, then for all $u \in [\tilde{t}_{i+1}, \tilde{t}_i]$,

$$\begin{aligned} &\left| e^{\bar{\lambda}^{\varepsilon,F}_{u,\tilde{t}_i}(\theta_0)} F(\theta(u), \varepsilon u) - e^{\bar{\lambda}^{\varepsilon,F_i}_{u,\tilde{t}_i}(\psi_i(0))} F(\psi_i(u), \varepsilon\tilde{t}_i) \right| \\ &\quad \leq \varepsilon \left(e^{\bar{\lambda}^{\varepsilon,F}_{u,\tilde{t}_i}(\theta_0)} E_1 + |F(\psi_i(u), \varepsilon\tilde{t}_i)| \varepsilon_0^{-1} \, e^{K M_1 + 1} \right) \\ &\quad \leq \varepsilon (e^{K M_1} E_1 + M \varepsilon_0 \, e^{K M_1 + 1}) =: \varepsilon E_2. \end{aligned}$$

So,

$$\tfrac{1}{\varepsilon} |d_i - \varepsilon k(\varepsilon\tilde{t}_i) F(\theta(\tilde{t}_i), \varepsilon\tilde{t}_i)| \leq \varepsilon k(\varepsilon\tilde{t}_i) E_2$$

and hence,

$$\tfrac{1}{\varepsilon} \sum_{i=1}^{N-1} |d_i - \varepsilon k(\varepsilon\tilde{t}_i) F(\theta(\tilde{t}_i), \varepsilon\tilde{t}_i)| \leq E_2.$$

Thus, overall, Eq. (A5) is satisfied with

$$E = R(K M \, e^{K M_1} + E_2).$$

□

Let us note that the bound E obtained in the above proof can probably be considerably improved through more detailed calculation and less estimation; but we have obtained a cruder bound for the sake of a faster proof.

APPENDIX B: NUMERICS

Throughout this paper, additively forced Adler equations $\dot{\theta}(t) = -a \sin(\theta(t)) + G(t)$ were simulated by numerical integration using a fourth order Runge–Kutta scheme, with a time step of 0.01 s. The FTLEs associated with trajectories $\theta(t)$ were calculated simply by taking the time-averaged value of $-a \cos(\theta(t))$ over the time-interval of interest, as in (11). To obtain the initial condition $\theta(0)$ for a given final state $\theta(T)$, the value of $\theta(T)$ was used as the initial condition of a forward-time simulation of the equation $\dot{\theta}(t) = a \sin(\theta(t)) - G(T - t)$.

In Sec. II B, the function $g(t)$ was constructed as follows. First, a sample path W_t of a zero-drift Brownian motion on $[0, \mathfrak{T}]$ of diffusion parameter $\sigma = \frac{1}{\sqrt{\mathfrak{T}}}$ was constructed by cumulative addition of Gaussian random increments over a time step of 0.01 s. Then, the Brownian bridge was constructed as $W_t - \frac{t}{\mathfrak{T}} W_{\mathfrak{T}}$. Then, the result was passed through a fifth order Butterworth low-pass filter with cut-off frequency $1/(2\pi \times 10^3)$ Hz, performed via cascaded second-order sections (in Python, with the function `scipy.signal.sosfilt`), and we linearly interpolated the output of the filter.

In Figs. 5 and 6, the ω-values indicated in red were numerically obtained as follows: For the unwrapped phase $x(t)$ as governed by the differential equation

$$\dot{x}(t) = -a\sin(x(t)) + k + A\cos(\omega t)$$

on the real line, setting $x(0) = 0$, it was observed by simulation that $x\left(\frac{2\pi}{\omega}\right)$ increased approximately linearly with $1/\omega$, with increments across consecutive values in the $(1/\omega)$-discretization being strictly positive and very small compared to 2π. Hence, it was possible to carry out linear interpolation of the simulated wrapped phase $\theta\left(\frac{2\pi}{\omega}\right)$ as a function of $1/\omega$ [with $\theta(0) = 0$]. Where this linearly interpolated function of $1/\omega$ crossed $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ is where the ω-values were marked; as in Fig. 5(c), the locations of $\theta\left(\frac{2\pi}{\omega}\right)$ are indistinguishably the same for the other 49 initial conditions $\theta(0) = \frac{2\pi i}{50}$ as for $\theta(0) = 0$.

The basis for this procedure is as follows: since $k > a$ and $A \in (k - a, k + a)$, Proposition 41(B) can be applied to the time-0-to-$\frac{2\pi}{\omega}$ mapping Φ of (23) to give that for all sufficiently small ω, Φ maps all points starting outside some small arc P into a small arc Q. By the symmetries of (23), we have that Q is close to P if and only if Q is close to either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. Now, on the one hand, when P and Q are not close to each other, it is clear by considering the graph of Φ that Φ has a stable fixed point near Q and an unstable fixed point near P, and thus system (23) is exponentially stable in the sense of Definition 35. However, on the other hand, if Q and P cross past each other (which, again, is equivalent to Q crossing past either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$) as ω is varied then, again by considering the graph of Φ, during such crossing there must occur an interval of ω-values for which Φ has no fixed points and, therefore, by Proposition 37, the system (23) is neutrally stable in the sense of Definition 33 (see also Ref. 17).

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