Journal of Symbolic Computation 104 (2021) 724–753

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Computing invariants for multipersistence via spectral systems and effective homology

Andrea Guidolin [a], Jose Divasón [b], Ana Romero [b], Francesco Vaccarino [c]

[a] *Basque Centre for Applied Mathematics - Bilbao, Spain*
[b] *Universidad de La Rioja - Logroño, Spain*
[c] *Politecnico di Torino & ISI Foundation - Turin, Italy*

ARTICLE INFO

Article history:
Available online 29 September 2020

Keywords:
Symbolic computation
Constructive algebraic topology
Multipersistence
Spectral systems
Effective homology

ABSTRACT

Both spectral sequences and persistent homology are tools in algebraic topology defined from filtrations of objects (e.g. topological spaces or simplicial complexes) indexed over the set $\mathbb{Z}$ of integer numbers. A recent work has shown the details of the relation between both concepts. Moreover, generalizations of both concepts have been proposed which originate from a different choice of the set of indices of the filtration, producing the new notions of *multipersistence* and *spectral system*. In this paper, we show that these notions are also related, generalizing results valid in the case of filtrations over $\mathbb{Z}$. By using this relation and some previous programs for computing spectral systems, we have developed a new module for the Kenzo system computing multipersistence. We also present a birth-death descriptor and a new invariant providing information on multifiltrations. This new invariant, in some cases, is able to provide more information than the rank invariant. We show some applications of our algorithms to spaces of infinite type via the effective homology technique, where the performance has also been improved by means of discrete vector fields.

E-mail addresses: aguidolin@bcamath.org (A. Guidolin), jose.divason@unirioja.es (J. Divasón), ana.romero@unirioja.es (A. Romero), francesco.vaccarino@polito.it (F. Vaccarino).

https://doi.org/10.1016/j.jsc.2020.09.007

1. Introduction

Persistent homology (Edelsbrunner et al. (2002); Zomorodian and Carlsson (2005)) is a technique in computational algebraic topology conceived to summarize the information of a filtration (usually of simplicial complexes) in the form of topological invariants. Homology is used to study the topological features at each point of the filtration and to track their evolution across the whole filtration. Since simplicial complexes are in many situations convenient objects to be associated with data of different type (e.g. point clouds, networks, digital images), persistent homology represents a versatile method for the analysis of data, which significantly contributed to the development of *topological data analysis*.

Spectral sequences (McCleary (2001)) are a tool in algebraic topology which provides information on the homology of a complex by means of successive approximations and are also defined by means of filtrations. The notions of persistent homology and spectral sequence are related, as explained in Basu and Parida (2017) using exact couples, a classical construction in algebraic topology, complementing a previous approach (Romero et al. (2014)).

In their original setting, both spectral sequences and persistent homology are defined from filtrations with indices in $\mathbb{Z}$. Nevertheless, generalizations of both concepts have been proposed which originate from a different choice of the index set of the filtration. Multipersistence (Carlsson and Zomorodian (2009)) is a generalization of persistent homology for filtrations with indices in $\mathbb{Z}^m$. On the other hand, spectral sequences have been generalized in Matschke (2013) to the case of filtrations over any partially ordered set, producing the notion of spectral system.

As said, the goal of persistent homology is to provide an *invariant* which summarizes the topological properties of a filtration. Persistence diagrams and the barcode are used in persistent homology to capture the birth and death of homology classes across the filtration, and are *complete* invariants: they completely characterize (up to isomorphism) an algebraic structure associated with the filtration, called persistence module. Since there is no complete invariant in the multiparameter case (Carlsson and Zomorodian (2009)), incomplete invariants are defined for multipersistence. The most common one is the rank invariant, which provides useful information, but it sometimes cannot distinguish between non-isomorphic persistence modules. Moreover, detecting births and deaths of homology classes is much more complicated in the multiparameter case.

The main contributions of this work are as follows. We first show the relation between generalized persistent homology and spectral systems in a general scenario. Then, we propose a new implementation of programs for multipersistence as a new module for the computer algebra system Kenzo (Dousson et al. (1999)), based on our programs for spectral systems (Guidolin and Romero (2018)). In addition to computing well-studied invariants of multipersistence in a new way, which differentiates in some key aspects from all the available implementations, we propose and compute both a new descriptor and a new invariant for multipersistence. Our programs make use of the effective homology technique (Rubio and Sergeraert (2002)), which allows to computationally handle infinitely generated objects, extending in this way the domain of applicability of our algorithms. As far as we know, this feature has never been exploited in any other multipersistence software. Furthermore, we use discrete vector fields (Forman (1998)) to improve the programs and we provide examples of applications.

The programs have been implemented as a new module for the computer algebra system Kenzo and are available at:

https://github.com/ana-romero/Kenzo-external-modules

This work presents a revised and extended version of our previous conference paper (Guidolin et al. (2019)). All the sections include more details, explanations and proofs. Major changes and extensions are present in the following sections:

- Section 3 has been entirely rewritten to provide more details and results which clarify the relation between spectral systems and multipersistence.
- Section 5 has been reworked to clarify the intuitive idea behind the descriptor we proposed in Guidolin et al. (2019), confusingly termed invariant therein. The descriptor has been conceived

to extract birth-death information from multiparameter filtrations. We extend the work by defining a new invariant by means of the spectral system associated with a filtration, and we show its similarities with the birth-death descriptor, including examples of computations in Kenzo for both notions.
- We have added a new section, cf. Section 8, where we present some detailed examples of applications of our algorithms implemented in the Kenzo system.

The paper is structured as follows. In the next section we present preliminary results about multipersistence and spectral systems that we need for our constructions. Section 3 shows the relationship between both concepts. The generalization of the rank invariant and its computation in the case of finitely generated modules are presented in Section 4. The descriptor, which provides information about the births and deaths, and the new invariant defined from spectral system are introduced in Section 5. Section 6 shows how the effective homology technique is used in our programs to deal with computations involving infinitely generated spaces. We describe how discrete vector fields are used to improve the algorithms in Section 7. Then, we show some examples of applications and computations in Section 8. Finally, we present a summary and possible further work in Section 9.

2. Preliminaries

2.1. Multipersistence

In order to introduce *multipersistence*, also called *multiparameter* or *multidimensional persistence* by some authors, let us first illustrate some fundamental concepts of persistent homology theory over a fixed field $\mathbb{F}$. For more details and examples of applications we refer the reader to the surveys Kerber (2016); Edelsbrunner and Morozov (2012).

A *finite* $\mathbb{Z}$*-filtration* of a simplicial complex K is a sequence of subcomplexes

$$\emptyset = \ldots = K_{-1} = K_0 \subseteq \ldots \subseteq K_p \subseteq K_{p+1} \subseteq \ldots \subseteq K_N = K_{N+1} = \ldots = K.$$

Geometrical intuition is helpful to understand how the homology groups $H_n(K)$, and in particular the Betti numbers $\beta_n := \dim_{\mathbb{F}} H_n(K)$, describe the topological properties of K. Intuitively, we can say that β_n counts n-dimensional holes of K: β_0 is the number of connected components, β_1 the number of "tunnels", β_2 the number of "voids", and so on. The general idea of persistent homology is then to detect, using homology, the topological features which "persist" across the filtration. In order to do this, for every pair of indices $s \leq t$ in the filtration consider the map $f_n^{s,t} : H_n(K_s) \to H_n(K_t)$ induced in homology by the inclusion of simplicial complexes $K_s \hookrightarrow K_t$.

Definition 1. For every pair of indices $s \leq t$ we define a *persistent n-homology group* $H_n^{s,t}(K)$ as the subspace of $H_n(K_t)$ given by the image of the map $f_n^{s,t}$:

$$H_n^{s,t}(K) := \operatorname{Im}(f_n^{s,t} : H_n(K_s) \to H_n(K_t)).$$

We denote its dimension (as $\mathbb{F}$-vector space) $\beta_n^{s,t} := \dim_{\mathbb{F}} H_n^{s,t}(K)$, called a *persistent Betti number*.

One says that a homology class *is born* at time $i \in \mathbb{Z}$ if it is an element of $H_n(K_i)$ not belonging to the image $\operatorname{Im} f_n^{i-1,i}$. A homology class in $H_n(K_{j-1})$ is then said to *die* at time $j \in \mathbb{Z}$ if its image under $f_n^{j-1,j}$ is zero, otherwise it is said to *persist*; the homology classes which persist until the last step $N \in \mathbb{Z}$ of the filtration are said to *live forever*. Note that for this intuition to be rigorous one has to fix bases of the vector spaces $H_n(K_i)$ in accordance with the Fundamental Theorem of Persistent Homology (Zomorodian and Carlsson (2005)): see (Otter et al., 2017, Remark 5). Using this terminology, it is easy to see that for all $i < j$ the non-negative integer

$$\mu_n^{i,j} := (\beta_n^{i,j-1} - \beta_n^{i,j}) - (\beta_n^{i-1,j-1} - \beta_n^{i-1,j}) \tag{1}$$

is the number of distinct n-homology classes that are born at time i and die at time j. As first observed in Zomorodian and Carlsson (2005), the collection $\{\beta_n^{i,j}\}$ of persistent Betti number is a *complete* topological invariant, intuitively meaning that it captures all the topological information of a filtration. This notion can be made precise by introducing *persistence modules* (see Zomorodian and Carlsson (2005)): considering their decomposition as $\mathbb{F}[x]$-modules, it can be proven that two persistence modules are isomorphic if and only if they are described by the same collection of persistent Betti numbers. This invariant is sometimes represented in the equivalent form of a *persistence diagram* or a *barcode* for more effective visualization.

In some applications a setting in which simplicial complexes vary according to two or more parameters may be more interesting, for example because the interplay of the parameters can reveal information on the data. Combining the different parameters, one can build a filtration along m axes, which potentially encodes much more information than m linear filtrations considered one at a time.

Definition 2. Consider $\mathbb{Z}^m$, endowed with the usual coordinate-wise partial order $\leq$. A collection of simplicial complexes $(K_v)_{v\in\mathbb{Z}^m}$ such that $K_v \subseteq K_w$ if $v \leq w$ is called a $\mathbb{Z}^m$*-filtration* of simplicial complexes.

Definition 3. A $\mathbb{Z}^m$-filtration $(K_v)_{v\in\mathbb{Z}^m}$ of simplicial complexes is *finite* if there exists an element $w = (w_1, \dots, w_m) \in \mathbb{Z}^m$ such that, for each $i = 1, \dots, m$, each $\mathbb{Z}$-filtration obtained fixing $m-1$ parameters except the i-th, here denoted $(\hat{K}_p^{(i)})_{p\in\mathbb{Z}}$, is finite, with

$$\emptyset = \ldots = \hat{K}_{-1}^{(i)} = \hat{K}_0^{(i)} \subseteq \hat{K}_1^{(i)} \subseteq \ldots \subseteq \hat{K}_{w_i}^{(i)} = \hat{K}_{w_i+1}^{(i)} = \ldots$$

Multipersistence (Carlsson and Zomorodian (2009)) is a generalization of persistent homology which deals with $\mathbb{Z}^m$-filtrations instead of usual $\mathbb{Z}$-filtrations. The purpose is (again) to use homology to describe the evolution of topological features across a $\mathbb{Z}^m$-filtration of simplicial complexes. As we have seen, the ultimate goal of persistent homology is to provide an *invariant*, an object associated with a filtration which summarizes its topological properties. Unlike the 1-parameter case, there is no discrete complete invariant for multiparameter persistence. To support this claim, relying again on the concept of persistence module, one can endow the homology of a $\mathbb{Z}^m$-filtration with the structure of a $\mathbb{F}[x_1, \dots, x_m]$-module, and consider that the classification of $\mathbb{F}[x_1, \dots, x_m]$-modules is known to be very hard for $m > 1$. The impossibility to produce a complete invariant in the multiparameter case has been proved in Carlsson and Zomorodian (2009) through algebraic geometry arguments, but more recently also arguments from quiver representation theory have been proposed (see for instance Oudot (2015)). Nevertheless, invariants can be defined for multipersistence which are informative and relatively easy to compute. One of the most relevant in applications is the *rank invariant*, an immediate generalization of persistent Betti numbers proposed in Carlsson and Zomorodian (2009).

Definition 4. Let $(K_v)_{v\in\mathbb{Z}^m}$ be a $\mathbb{Z}^m$-filtration of simplicial complexes and let $v \leq w$ in $\mathbb{Z}^m$. We denote $f_n^{v,w} : H_n(K_v) \to H_n(K_w)$ the map induced in homology by the inclusion $K_v \hookrightarrow K_w$ and define

$$\beta_n^{v,w} := \dim_{\mathbb{F}} \operatorname{Im}(f_n^{v,w} : H_n(K_v) \to H_n(K_w)).$$

The collection of all $\beta_n^{v,w}$, for every pair of indices $v \leq w$ and for every n, is called *rank invariant* of the $\mathbb{Z}^m$-filtration.

Even if in the present work we will focus mainly on the rank invariant, we want to recall that other invariants have been proposed for multipersistence (Carlsson et al. (2010); Chacholski et al. (2017); Cerri et al. (2013); Lesnick and Wright (2015); Vaccarino et al. (2017); Scolamiero et al. (2017); Harrington et al. (2019); Dey and Xin (2018); Miller (2017)).

It is worth recalling the definition one-critical filtrations, introduced in Carlsson et al. (2009), which are very commonly used in multipersistence since they yield "better-behaved" persistence modules. A finite $\mathbb{Z}^m$-filtration of simplicial complexes $(K_v)_{v\in\mathbb{Z}^m}$ is called *one-critical* if, for every simplex σ, there exists exactly one filtration degree $v = v(\sigma) \in \mathbb{Z}^m$ such that $\sigma \in K_v - \bigcup_{u<v} K_u$. Given a finite set of points $\mathcal{X} := \{X^{(1)}, \ldots, X^{(k)}\} \subseteq \mathbb{Z}^m$, where $X^{(j)} = (x_1^{(j)}, \ldots, x_m^{(j)})$ for each $j \in \{1, \ldots, k\}$, we denote by glb the *greatest lower bound*, that can be explicitly expressed as

$$\mathrm{glb}(\mathcal{X}) = (\min_j x_1^{(j)}, \ldots, \min_j x_m^{(j)}).$$

One-critical filtrations have the following property:

Fact 5. Let (K_v) be a one-critical $\mathbb{Z}^m$-filtration of simplicial complexes, let $\mathcal{X} := \{X^{(1)}, \ldots, X^{(k)}\} \subseteq \mathbb{Z}^m$ and $Y := \mathrm{glb}(\mathcal{X})$. It holds

$$\bigcap_{j=1}^{k} K_{X^{(j)}} = K_Y.$$

Let us remark that clearly every $\mathbb{Z}^m$-filtration of simplicial complexes (K_v) determines a $\mathbb{Z}^m$-filtration of chain complexes (F_v), where F_v denotes $C_*(K_v)$. Although we have chosen to introduce persistence theory and multipersistence using filtrations of simplicial complexes, in the following sections we will consider the more general framework of filtered chain complexes. Moreover, besides $\mathbb{Z}^m$-filtration, in this work we will consider filtrations indexed over any partially ordered set (*poset*):

Definition 6. A filtration of a chain complex $C_* = (C_n, d_n)$ over a poset $(I, \leq)$, briefly called an *I-filtration*, is a collection of chain subcomplexes $F = (F_iC_*)_{i\in I}$ such that $F_iC_* \subseteq F_jC_*$ whenever $i \leq j$ in I. We will often denote the chain subcomplexes simply as F_i, forgetting about the grading of homology, when we are only interested in the filtration index i.

We now briefly review some key definitions of persistence theory in the general framework of the present paper, namely assuming that indices are elements of a poset $(I, \leq)$. A *persistence module* $\mathbb{V}$ is a collection of vector spaces $\{V_i\}_{i\in I}$ and linear maps $\{\ell_i^j : V_i \to V_j\}_{i\leq j}$ such that $\ell_i^i = \mathrm{Id}$, for all $i \in I$, and $\ell_j^k \ell_i^j = \ell_i^k$, for all $i \leq j \leq k$. A *morphism* between the persistence modules $\mathbb{V} = \{V_i, \ell_i^j : V_i \to V_j\}$ and $\mathbb{U} = \{U_i, h_i^j : U_i \to U_j\}$ is a collection of linear maps $\varphi = \{\varphi_i : V_i \to U_i\}_{i\in I}$ such that $h_i^j \varphi_i = \varphi_j \ell_i^j$, for all $i \leq j$. If each φ_i is an isomorphism, φ is called an *isomorphism* of persistence modules. An *invariant* is a property preserved by isomorphism of persistence modules. Notice that we are always thinking of a persistence module $\mathbb{V}$ as obtained by applying n-homology to a filtration $(F_i)_{i\in I}$ of chain complexes (or simplicial complexes), so that $V_i = H_n(F_i)$ and the maps $\ell_i^j : H_n(F_i) \to H_n(F_j)$ are induced by inclusions $F_i \hookrightarrow F_j$. When the poset of indices is $\mathbb{Z}$ these definitions give us the (single-parameter) persistent homology case; when the poset of indices is $\mathbb{Z}^m$ for $m \geq 2$ these definitions describe the multipersistence case.

2.2. Spectral systems

Spectral systems are a construction that extends the classical definition of spectral sequence (McCleary (2001)) to the case of filtrations indexed over a partially ordered set (*poset*). Firstly, recall that for classical spectral sequences, which arise from a $\mathbb{Z}$-filtration $(F_p)_{p\in\mathbb{Z}}$, we have the formula (see MacLane (1963)):

$$E_{p,q}^r = \frac{Z_{p,q}^r + F_{p-1}C_{p+q}}{d(Z_{p+r-1,q-r+2}^{r-1}) + F_{p-1}C_{p+q}},$$

where $Z_{p,q}^r = \{a \in F_pC_{p+q} : d(a) \in F_{p-r}C_{p+q-1}\}$ and the usual convention is to denote $n := p + q$. This formula can be therefore rewritten as

$$E^r_{p,q} = \frac{F_pC_n \cap d^{-1}(F_{p-r}C_{n-1}) + F_{p-1}C_n}{F_pC_n \cap d(F_{p+r-1}C_{n+1}) + F_{p-1}C_n},$$

where we can see more clearly the interplay of 4 filtration indices: $p-r$, $p-1$, p and $p+r-1$.

In Matschke (2013), this formula was imitated and generalized to the case of I-filtrations by defining, for every 4-tuple of indices $z \le s \le p \le b$ in I, the *term*

$$S_n[z, s, p, b] := \frac{F_pC_n \cap d^{-1}(F_zC_{n-1}) + F_sC_n}{F_pC_n \cap d(F_bC_{n+1}) + F_sC_n}. \tag{2}$$

The collection of all such terms is called a *generalized spectral sequence* or a *spectral system* for the I-filtration $(F_i)_{i\in I}$. To gain familiarity with the definition, let us remark that the homology $H_n(F_p)$ of a chain subcomplex $F_p = F_pC_*$ can be expressed as $S_n[-\infty, -\infty, p, p]$, for each $p \in I$, with the convention that $F_{-\infty} = 0$. Similarly, for each $s \le p$ in I, the relative homology $H_n(F_p/F_s)$ can be expressed as $S_n[s, s, p, p]$. In the case of a $\mathbb{Z}$-filtration $(F_p)_{p\in\mathbb{Z}}$, the term $E^r_{p,q}$ of the associated spectral sequence can be expressed as $S_n[p-r, p-1, p, p+r-1]$.

In the rest of this section we include some results and definitions about spectral systems present in Matschke (2013) that we will use in our work. It is worth noting that Matschke's results are stated in a more general situation than the one presented above (his work is based on spectral systems which are not necessarily associated to a generalized filtration, but are defined in a more general structure named exact couples systems, see Section 3). Thus, with the aim of making this work self-contained, we have incorporated our own proofs of such results, which are more direct than the ones present in the original paper, since we can restrict ourselves to the particular case of spectral systems associated with a generalized filtration.

First of all, let us recall some well-known results on modules which will be used repeatedly in the proofs. In the next statements we suppose we have fixed a commutative ring R (in the present work, R is $\mathbb{Z}$ or a field $\mathbb{F}$), and we use the term module to mean R-module.

Fact 7. The following holds:

1. (The *modular law*). Let N, S, T be submodules of a module M. If $T \subseteq N$, then $N \cap (S+T) = N \cap S + T$.
2. Let $f : M \to M'$ be a morphism of modules and let N be a submodule of M such that $N \subseteq \operatorname{Ker} f$. Then there exists a unique morphism of modules $\varphi : M/N \to M'$ such that $\varphi q = f$, where q denotes the canonical projection $M \to M/N$. Furthermore, $\operatorname{Im}\varphi = \operatorname{Im} f$ and $\operatorname{Ker}\varphi = \operatorname{Ker} f/N$.
3. Let M be a module, and let S, T be submodules of M. Then $(S+T)/T \cong S/(S\cap T)$.
4. Let $T \subseteq S \subseteq M$ be modules. Then $(M/T)/(S/T) \cong M/S$.

The modular law is stated for example in MacLane (1963, p. 318), the other statements are proven in Atiyah and Macdonald (1969, Ch. 2). Using *1.* and *3.* of Fact 7 we can express the generic term (2) of a spectral system as

$$S_n[z, s, p, b] \cong \frac{F_pC_n \cap d^{-1}(F_zC_{n-1})}{F_pC_n \cap d(F_bC_{n+1}) + F_sC_n \cap d^{-1}(F_zC_{n-1})}, \tag{3}$$

which will be sometimes convenient in what follows. Notice that the submodule $F_pC_n \cap d(F_bC_{n+1})$ at the denominator can be written also as $d(F_bC_{n+1} \cap d^{-1}(F_pC_n))$.

To simplify notations, in the remainder of this section we denote F_iC_n simply by F_i (the appropriate degree n is always clear from the context), and denote canonical isomorphisms by the equal sign.

Lemma 8 *(Matschke (2013)). Let $z_1 \le s_1 \le p_1 \le b_1$ and $z_2 \le s_2 \le p_2 \le b_2$ be two 4-tuples of indices in I with $z_1 \le z_2$, $s_1 \le s_2$, $p_1 \le p_2$ and $b_1 \le b_2$. Then the inclusion of chain subcomplexes induces a well-defined map*

$$\ell : S_n[z_1, s_1, p_1, b_1] \to S_n[z_2, s_2, p_2, b_2]$$

for all n.

Proof. Express both $S_n[z_1, s_1, p_1, b_1]$ and $S_n[z_2, s_2, p_2, b_2]$ as in (3) and consider their numerators. Since $z_1 \leq z_2$ and $p_1 \leq p_2$, we have an inclusion $j : F_{p_1} \cap d^{-1}(F_{z_1}) \hookrightarrow F_{p_2} \cap d^{-1}(F_{z_2})$. Now, consider the canonical projections $q_1 : F_{p_1} \cap d^{-1}(F_{z_1}) \to S_n[z_1, s_1, p_1, b_1]$ and $q_2 : F_{p_2} \cap d^{-1}(F_{z_2}) \to S_n[z_2, s_2, p_2, b_2]$ and the composition $f := q_2 j$. We have

$$\operatorname{Ker} f = F_{p_1} \cap d^{-1}(F_{z_1}) \cap (F_{p_2} \cap d(F_{b_2}) + F_{s_2} \cap d^{-1}(F_{z_2})).$$

Since $z_1 \leq z_2$, $s_1 \leq s_2$, $p_1 \leq p_2$ and $b_1 \leq b_2$, the denominator $N_1 := F_{p_1} \cap d(F_{b_1}) + F_{s_1} \cap d^{-1}(F_{z_1})$ of $S_n[z_1, s_1, p_1, b_1]$ is a submodule of $\operatorname{Ker} f$, so ℓ is well-defined as the unique morphism such that $\ell q_1 = f$. $\square$

For the sake of readability, we avoid decorating the maps ℓ with indices denoting the domain, the codomain and the degree n. Observe that, with the notations of the previous proof, $\operatorname{Ker} \ell = \operatorname{Ker} f / N_1$ and

$$\operatorname{Im} \ell = \operatorname{Im} f = \frac{F_{p_1} \cap d^{-1}(F_{z_1}) + N_2}{N_2}, \tag{4}$$

with $N_2 := F_{p_2} \cap d(F_{b_2}) + F_{s_2} \cap d^{-1}(F_{z_2})$.

Lemma 9 *(Matschke (2013)). For any $z \leq p_1 \leq p_2 \leq p_3 \leq b$ in I, the maps induced by inclusions produce a short exact sequence*

$$0 \to S_n[z, p_1, p_2, b] \xrightarrow{\ell} S_n[z, p_1, p_3, b] \xrightarrow{\ell'} S_n[z, p_2, p_3, b] \to 0, \tag{5}$$

for all n.

Proof. We prove exactness at the middle term. Using the explicit formulas for $\operatorname{Ker} \ell'$ and $\operatorname{Im} \ell$ stated above, we see that

$$\frac{\operatorname{Ker} \ell'}{\operatorname{Im} \ell} \cong \frac{F_{p_3} \cap d^{-1}(F_z) \cap (F_{p_3} \cap d(F_b) + F_{p_2} \cap d^{-1}(F_z))}{F_{p_3} \cap d^{-1}(F_z) + F_{p_2} \cap d^{-1}(F_z)} \cong 0.$$

Similarly, using the explicit formulas for $\operatorname{Ker} \ell$ and $\operatorname{Im} \ell'$ one can easily check that ℓ is injective and ℓ' is surjective. $\square$

Lemma 10 *(Matschke (2013)). Given an I-filtration $(F_i)_{i \in I}$ for a chain complex C_* and three 4-tuples of indices satisfying the condition*

$$\begin{array}{ccccccccccc}
 & & & & z_3 & \leq & s_3 & \leq & p_3 & \leq & b_3 \\
 & & & & \| & & \| & & & & \\
 & & z_2 & \leq & s_2 & \leq & p_2 & \leq & b_2 & & \\
 & & \| & & \| & & & & & & \\
z_1 & \leq & s_1 & \leq & p_1 & \leq & b_1 & & & &
\end{array}$$

the differential of the chain complex C_* induces differentials d_3, d_2 between the terms

$$S_{n+1}[z_3, s_3, p_3, b_3] \xrightarrow{d_3} S_n[z_2, s_2, p_2, b_2] \xrightarrow{d_2} S_{n-1}[z_1, s_1, p_1, b_1]$$

and by taking homology we obtain

$$\frac{\operatorname{Ker} d_2}{\operatorname{Im} d_3} \cong S_n[s_1, s_2, p_2, p_3].$$

Proof. We adapt the arguments of Weibel (1994, Construction 5.4.6) for classical spectral sequences to our current choice of indices in I. To show that the induced differentials are well-defined, let us focus on $d_2 : S_n[z_2, s_2, p_2, b_2] \to S_{n-1}[z_1, s_1, p_1, b_1]$. Similarly to the proof of Lemma 8, express both $S_n[z_2, s_2, p_2, b_2]$ and $S_{n-1}[z_1, s_1, p_1, b_1]$ as in (3) and consider their numerators. Since $z_2 = p_1$, the image $d(F_{p_2} \cap d^{-1}(F_{z_2})) = F_{z_2} \cap d(F_{p_2})$ is contained in $F_{p_1} \cap d^{-1}(F_{z_1})$. Let g be the morphism

$$F_{p_2} \cap d^{-1}(F_{z_2}) \xrightarrow{d} F_{z_2} \cap d(F_{p_2}) \hookrightarrow F_{p_1} \cap d^{-1}(F_{z_1}) \xrightarrow{q_1} S_{n-1}[z_1, s_1, p_1, b_1],$$

where q_1 is the canonical projection. We have

$$\begin{aligned}\operatorname{Ker} g &= \{x \in F_{p_2} \cap d^{-1}(F_{z_2}) \mid d(x) \in F_{p_1} \cap d(F_{b_1}) + F_{s_1} \cap d^{-1}(F_{z_1})\} \\ &= F_{b_1} \cap d^{-1}(F_{p_1}) + F_{p_2} \cap d^{-1}(F_{s_1}).\end{aligned}$$

Since $z_2 = p_1$ and $s_2 = b_1$, the denominator $N := F_{p_2} \cap d(F_{b_2}) + F_{s_2} \cap d^{-1}(F_{z_2})$ of $S_n[z_2, s_2, p_2, b_2]$ is contained in $\operatorname{Ker} g$. As a result, the map d_2 is well-defined on the quotient.

Let us prove now the last part of the claim. Clearly, $\operatorname{Ker} d_2 = \operatorname{Ker} g/N$ and $\operatorname{Im} d_3 = (F_{z_3} \cap d(F_{p_3}) + N)/N$, so

$$\begin{aligned}\frac{\operatorname{Ker} d_2}{\operatorname{Im} d_3} &\cong \frac{F_{p_2} \cap d^{-1}(F_{s_1}) + F_{b_1} \cap d^{-1}(F_{p_1})}{F_{z_3} \cap d(F_{p_3}) + F_{s_2} \cap d^{-1}(F_{z_2})} \cong \frac{F_{p_2} \cap d^{-1}(F_{s_1})}{F_{p_2} \cap d(F_{p_3}) + F_{s_2} \cap d^{-1}(F_{s_1})} \\ &\cong S_n[s_1, s_2, p_2, p_3] \quad \square\end{aligned}$$

Notice that generalized spectral sequences are in many aspects similar to classical ones. For example, Lemma 10 extends what in the classical case is the process of obtaining terms of the page $r + 1$ by taking homology at page r.

The paper Matschke (2013) includes some explicit examples of spectral systems which generalize for instance the classical spectral sequences of Serre, Eilenberg–Moore and Adams–Novikov. However, as in the case of spectral sequences associated with a linear filtration, no algorithm is provided to compute the different components when the initial chain complexes are not finitely generated. Thanks to the method of *effective homology* (Rubio and Sergeraert (2002)), in Guidolin and Romero (2018) an algorithm is developed for computing spectral systems of spaces (possibly) of infinite type; the special case of the Serre spectral system is treated in Guidolin and Romero (2020). The corresponding programs were implemented as a new module for the system Kenzo (Dousson et al. (1999)), a symbolic computation software written in Common Lisp and devoted to algebraic topology, solving in this way also the classical problems of spectral sequences: determining differential maps and extensions. The effective homology method was also used by the third author in Romero et al. (2006) for computing spectral sequences in the case of $\mathbb{Z}$-filtrations.

3. Relation between spectral systems and multipersistence

In Basu and Parida (2017), a relation between spectral sequences and persistent homology (both defined for $\mathbb{Z}$-filtrations and taking homology over a fixed field $\mathbb{F}$) is proved by means of the classical notion of *exact couples* introduced in Massey (1952). Exact couples are collections of long exact sequences, with an additional hypothesis on the involved modules, which can be *derived* to obtain new exact sequences.

More precisely, each exact couple is a 5-tuple $(D^r, E^r, i^r, j^r, k^r)$, where D^r, E^r are $\mathbb{F}$-vector spaces and i^r, j^r, k^r are linear maps such that the triangular diagram

$$\begin{tikzcd} D^r \arrow[rr, "i^r"] & & D^r \arrow[dl, "j^r"] \\ & E^r \arrow[ul, "k^r"] & \end{tikzcd} \tag{6}$$

is exact at each vertex: $\operatorname{Ker}(j^r) = \operatorname{Im}(i^r)$, $\operatorname{Ker}(k^r) = \operatorname{Im}(j^r)$ and $\operatorname{Ker}(i^r) = \operatorname{Im}(k^r)$.

Associated with a $\mathbb{Z}$-filtration (F_p) there are exact couples $(D^r, E^r, i^r, j^r, k^r)$ where, for each $r \geq 1$, $D^r = \bigoplus_{p,q} H^{p,p+r-1}_{p+q}$ and $E^r = \bigoplus_{p,q} E^r_{p,q}$, with the linear maps i^r, j^r induced by inclusions and the linear map k^r induced by the differential. Each exact couple $(D^r, E^r, i^r, j^r, k^r)$ consists of a collection of long exact sequences

$$\cdots \to H_n^{p-1,p+r-2} \xrightarrow{i^r} H_n^{p,p+r-1} \xrightarrow{j^r} E^r_{p,q} \xrightarrow{k^r} H_{n-1}^{p-r,p-1} \xrightarrow{i^r} H_{n-1}^{p-r+1,p} \to \cdots \tag{7}$$

Using the simple fact that in an exact sequence of finite dimensional $\mathbb{F}$-vector spaces

$$\cdots \xrightarrow{f} U \xrightarrow{g} V \xrightarrow{h} W \xrightarrow{l} \cdots \tag{8}$$

it holds $\dim_{\mathbb{F}} V = (\dim_{\mathbb{F}} U - \dim_{\mathbb{F}} \operatorname{Im} f) + (\dim_{\mathbb{F}} W - \dim_{\mathbb{F}} \operatorname{Im} l)$, in Basu and Parida (2017) the following relation is obtained:

$$\dim_{\mathbb{F}} E^r_{p,q} = \beta_n^{p,p+r-1} - \beta_n^{p-1,p+r-1} + \beta_{n-1}^{p-r,p-1} - \beta_{n-1}^{p-r,p}, \tag{9}$$

for all integers p, q, r with $r \geq 1$ and $n := p+q$. This relation can be inverted, to express every persistent Betti number $\beta_n^{s,t}$ as a combination of the dimensions $\dim_{\mathbb{F}} E^r_{p,q}$. The existence of these relations intuitively means that the collections of integers $\{\beta_n^{s,t}\}$ and $\{\dim_{\mathbb{F}} E^r_{p,q}\}$ carry the same amount of topological information about the filtration.

In this section we show how the relation (9) can be generalized for the case of filtrations over a poset I (with some additional hypotheses), a result contained in the PhD thesis Guidolin (2018). A relevant part of the work consists in rephrasing in our generalized setting the arguments of Basu and Parida (2017). First of all, let us denote by $-\infty$ the minimum of the poset I, which can be added "artificially" to the poset if needed, and let us suppose that $F_{-\infty} = 0$. This assumption is consistent with the fact that we are ultimately interested in finite filtrations, having zero chain groups for small enough filtration indices.

The notion of exact and derived couples is generalized for I-filtrations in Matschke (2013, Definition 4.1) and referred to by the expression *exact couple system*. An exact couple system is again a collection of particular long exact sequences, where now the involved spaces are indexed over the poset I. Incidentally, exact couple systems can be seen as a way to define spectral systems that is even more general than the one we presented in Section 2.2. For the scope of the present work, however, we only need a specific property that exact couple systems share with classical exact couples (Proposition 11 below), which intuitively consists in a method to produce, from a collection of long exact sequences, a new collection of long exact sequences.

Before employing some long exact sequences of terms of the spectral system to deduce the sought relation, we introduce some relevant definitions. Firstly, let us state the natural generalization of the rank invariant (Definition 4) that we will use in what follows. Given an I-filtration $(F_i)_{i\in I}$ and $v \leq w$ in I, we define

$$\beta_n(v, w) := \dim_{\mathbb{F}} \operatorname{Im}(\ell : H_n(F_v) \to H_n(F_w)),$$

where ℓ is the map induced by the inclusion $F_v \hookrightarrow F_w$; we call *rank invariant* the collection of all $\beta_n(v, w)$, for any n and any $v \leq w$. The map $\ell : H_n(F_v) \to H_n(F_w)$ has been denoted by $f_n^{v,w}$ in Section 2 in the case of $\mathbb{Z}^m$-filtration. In the general case of a spectral system associated with an I-filtered chain complex, however, we prefer to use the notation ℓ for all the maps induced by inclusion, which are well-defined whenever the assumptions of Lemma 8 hold.

Let us now introduce the class of posets of interest for the present section. A *partially ordered abelian group* $(I, +, \leq)$ is an abelian group $(I, +)$ endowed with a partial order $\leq$ that is *translation invariant*: for all $p, t, t' \in I$, if $t \leq t'$ then $p + t \leq p + t'$.

We now state a property of exact couples which we will apply to spectral systems associated with I-filtrations, with I a partially ordered abelian group.

Proposition 11 *(Massey (1952); McCleary (2001)). Let (D, E, i, j, k) be an exact couple. The map $\partial := jk$ is a differential $\partial : E \to E$ and there is an exact couple (D', E', i', j', k'), called the* derived couple, *such that $D' = \operatorname{Im} i$ and $E' = \operatorname{Ker} \partial / \operatorname{Im} \partial$.*

Remark 12. The maps i', j', k' of Proposition 11 are respectively induced by i, j, k. For more details, we address the reader to McCleary (2001, § 2.2). As we show below, our application to spectral systems over a partially ordered abelian group I consists in fixing $p, u \in I$, with $u \geq 0$, and considering, for some integer $r \geq 1$,

$$D := \bigoplus_{n,h \in \mathbb{Z}} S_n[-\infty, -\infty, p+hu, p+(h+r-1)u],$$

$$E := \bigoplus_{n,h \in \mathbb{Z}} S_n[p+(h-r)u, p+(h-1)u, p+hu, p+(h+r-1)u].$$

Given an I-filtration with I a partially ordered abelian group, consider at first a collection of "simple" long exact sequences of relative homology defined from the filtration, like in the case of exact couples:

$$\cdots \to S_n[-\infty, -\infty, s, s] \xrightarrow{\ell} S_n[-\infty, -\infty, p, p] \xrightarrow{\ell} S_n[s, s, p, p] \xrightarrow{k} S_{n-1}[-\infty, -\infty, s, s] \xrightarrow{\ell} S_{n-1}[-\infty, -\infty, p, p] \to \cdots$$

for each $s \leq p$ in I. Denoting $v := p - s$ and imitating the classical construction of the derived couple (Proposition 11 and Remark 12), we obtain long exact sequences

$$\cdots \to S_n[-\infty, -\infty, p-v, p] \xrightarrow{\ell} S_n[-\infty, -\infty, p, p+v] \xrightarrow{\ell} S_n[p-2v, p-v, p, p+v] \xrightarrow{k} S_{n-1}[-\infty, -\infty, p-2v, p-v] \xrightarrow{\ell} S_{n-1}[-\infty, -\infty, p-v, p] \to \cdots$$

where the involved vector spaces are determined as images $\operatorname{Im}\ell$ or applying Lemma 10. The maps denoted by ℓ are again induced by inclusion, and k is induced by the differential. This construction clearly can be iterated, yielding for each integer $r \geq 1$ long exact sequences of the form

$$\cdots \to S_n[-\infty, -\infty, p-v, p+(r-2)v] \xrightarrow{\ell} S_n[-\infty, -\infty, p, p+(r-1)v] \xrightarrow{\ell} S_n[p-rv, p-v, p, p+(r-1)v] \xrightarrow{k} S_{n-1}[-\infty, -\infty, p-rv, p-v] \xrightarrow{\ell} S_{n-1}[-\infty, -\infty, p-(r-1)v, p] \to \cdots.$$

With a slight modification of this argument, for any element $w \geq 0$ of I we can obtain long exact sequences of the form

$$\cdots \to S_n[-\infty, -\infty, p-v, p-v+w] \xrightarrow{\ell} S_n[-\infty, -\infty, p, p+w] \xrightarrow{\ell} S_n[p-v-w, p-v, p, p+w] \xrightarrow{k} S_{n-1}[-\infty, -\infty, p-v-w, p-v] \xrightarrow{\ell} S_{n-1}[-\infty, -\infty, p-w, p] \to \cdots. \quad (10)$$

Exactness can be proven either directly or considering the filtrations with indices $\ldots \leq p-v-w \leq p-v \leq p \leq p+w \leq \ldots$ Recalling equation (4), observe that if $p \leq u$ and $b \leq t$ in I then

$$S_n[-\infty, -\infty, p, t] = \operatorname{Im}(\ell : S_n[-\infty, -\infty, p, b] \to S_n[-\infty, -\infty, u, t]). \quad (11)$$

Since in particular $S_n[-\infty, -\infty, p, t] = \operatorname{Im}(\ell : H_n(F_p) \to H_n(F_t))$, which yields by definition $\dim_{\mathbb{F}} S_n[-\infty, -\infty, p, t] = \beta_n(p, t)$, from (10) and using the simple fact stated after (8) we can obtain the generalization we sought for.

Theorem 13. *Let $(S_n[z,s,p,b])$ be the spectral system associated with an I-filtration, with I a partially ordered abelian group. For all $p, v, w \in I$ such that $v, w \geq 0$, the dimension $\dim_{\mathbb{F}} S_n[p-v-w, p-v, p, p+w]$ is equal to*

$$\beta_n(p, p+w) - \beta_n(p-v, p+w) + \beta_{n-1}(p-v-w, p-v) - \beta_{n-1}(p-v-w, p). \tag{12}$$

Remark 14. Even if the choice of $p, v, w \in I$ in Theorem 13 allows to independently select only three out of the four indices of a term $S_n[z,s,p,b]$, for example z, p, b, we can intervene on the fourth index using Lemma 9: if $z \leq p_1 \leq p_2 \leq p_3 \leq b$ are indices in I, then

$$\dim_{\mathbb{F}} S_n[z, p_1, p_3, b] = \dim_{\mathbb{F}} S_n[z, p_1, p_2, b] + \dim_{\mathbb{F}} S_n[z, p_2, p_3, b].$$

Consider now a finite $\mathbb{Z}^m$-filtration of simplicial complexes $(K_v)_{v\in\mathbb{Z}^m}$ and the induced $\mathbb{Z}^m$-filtration of chain complexes $(F_v)_{v\in\mathbb{Z}^m}$. Theorem 13, together with Remark 14, allows to express the dimension of each term $S_n[z,s,p,b]$ of the associated spectral system as a combination of some $\beta_n^{v,w}$. Notice that, using a recursive argument, the relation (12) can be inverted, allowing to express every $\beta_n^{v,w}$ as a combination of the dimensions of some terms $S_n[z,s,p,b]$. This allows to conclude that the spectral system over $\mathbb{Z}^m$ carries the same amount of topological information on the filtration as the rank invariant of Definition 4, as the collections $\{\dim_{\mathbb{F}} S_n[z,s,p,b]\}$ and $\{\beta_n^{v,w}\}$ can be determined one from the other.

In Section 5.2 we present a second application of the previous argument, considering a different poset related to multipersistence.

4. Generalizing the rank invariant in the finite case

As we mentioned in Section 2, a number of invariants for multipersistence have been proposed, and a few implementations are available. Let us name some of them, addressing the interested reader to recent works like Dey and Xin (2019) for a more complete list of references. In Carlsson et al. (2010) the authors propose for the first time an efficient algorithm to compute invariants associated with resolutions of modules constructed from $\mathbb{Z}^m$-filtrations, although some restrictive assumptions are made on the type of filtrations; a more general framework is studied in Chacholski et al. (2017); efficient algorithms for 2-parameter persistence are presented in Lesnick and Wright (2019). In Cerri et al. (2013) the study of a $\mathbb{Z}^m$-filtration is reduced to a family of $\mathbb{Z}$-filtrations corresponding to linear sections with different slopes. This idea has been further developed in Lesnick and Wright (2015), together with the theoretical bases of the software RIVET for visualizing 2-parameter persistence. The paper Harrington et al. (2019) presents an interesting approach via commutative algebra. Efficient methods to deal with a particular class of 2-parameter persistence modules are introduced in Dey and Xin (2018). A different special class of 2-parameter persistence modules that admits a decomposition with “simple” indecomposables is studied in Cochoy and Oudot (2020). In Scolamiero et al. (2017) an algebraic definition of noise (negligible topological features) for multipersistence is introduced and some related invariants are studied. Real multipersistence modules are studied in Miller (2017); to this aim, downsets (see below) in $\mathbb{R}^m$ play a key role. Generalized persistent homology and its relation with filtrations of weighted graphs is studied in Vaccarino et al. (2017).

Trying to generalize the existing programs, each of which was developed to deal with particular situations, we propose a new implementation of multipersistence as a new module for the system Kenzo, making use of our previous programs for computing spectral systems presented in Guidolin and Romero (2018). Our new programs are in several respects more general than the existing ones, since they compute multipersistence over integer coefficients and they can be applied to filtrations over any poset. Moreover, as we will show in Section 6, thanks to the *effective homology* technique our algorithms can be used to determine multipersistence of spaces of infinite type, a unique feature among the available software for multipersistence. Our programs are written in the Common Lisp programming language, making use of functional programming to deal with infinitely generated spaces

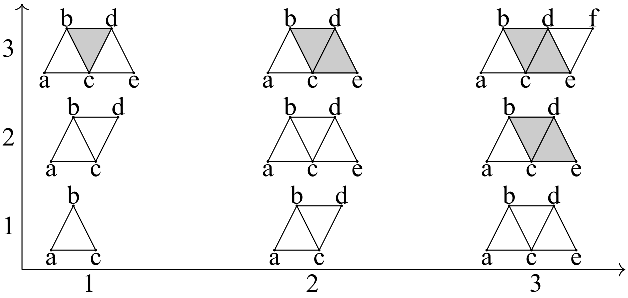

Fig. 1. Small simplicial complex filtered over $\mathbb{Z}^2$.

and general posets. The implementation of the effective homology technique makes our programs less efficient than available implementations for computations in multipersistence; however, this is a necessary trade-off for extending the domain of applicability of our algorithms to infinitely generated chain complexes and filtrations over general posets. For this reason, we do not include in the present work a computational efficiency comparison with other software, since our aim is to complement the available implementations with new and unique features rather than improve their computational speed.

Since we start from persistent homology groups and the rank invariant, we first extend the computation of these notions to the case of I-filtrations. Let $(F_i)_{i\in I}$ be an I-filtration of a chain complex C_* and $v \leq w$ in I. We consider the quotient group

$$H_n^{v,w} := \frac{F_v C_n \cap \operatorname{Ker} d_n}{F_v C_n \cap d(F_w C_{n+1})}, \tag{13}$$

called a *(generalized) persistent homology group*, which clearly represents the homology classes in $H_n(F_v)$ which are still present in $H_n(F_w)$, that is, it corresponds to $\operatorname{Im}(\ell : H_n(F_v) \to H_n(F_w))$. When computing this group with coefficients in a field, its rank corresponds to the rank invariant. In our case, we have developed a Kenzo function computing the group with integer coefficients, producing not only the rank but also the generators and the torsion coefficients. Our programs use some previous functions computing spectral systems developed in Guidolin and Romero (2018), since some of the subgroups appearing in the quotient (13) are similar to the subgroups appearing in the spectral system terms (2). Once these subgroups are determined, the corresponding quotient can be computed by means of diagonalization algorithms of matrices in a similar way to the algorithm used to compute homology groups by means of the Smith Normal Form technique (see Kaczynski et al. (2004)).

As a didactic example, let us consider the chain complex endowed with a (finite) $\mathbb{Z}^2$-filtration associated with the filtered simplicial complex of Fig. 1, which shows, corresponding to each of the points $(1,1)$, $(1,2)$, $(1,3)$, …, $(3,3) \in \mathbb{Z}^2$, a simplicial complex constituted by 0-simplices (points $a, b, c, \ldots$), 1-simplices (edges $ab, ac, \ldots$) and 2-simplices (the triangles bcd and cde). For example, in degree 1, there are two homology classes (1-dimensional holes) which live in $F_{(1,2)}$ and still live in $F_{(2,2)}$, so that $H_1^{(1,2),(2,2)} = \mathbb{Z}^2$, with generators given by the combinations $1 * ab - 1 * ac + 1 * bc$ and $-1 * ab + 1 * ac - 1 * bd + 1 * cd$. However, there is only one class which lives in $F_{(1,2)}$ and still lives in $F_{(3,3)}$, so that $H_1^{(1,1),(3,3)} = \mathbb{Z}$, generated in this case by the combination $1 * ab - 1 * ac + 1 * bc$. The second class has *died* because the triangle bcd has been filled.

```
> (multiprst-group K '(1 2) '(2 2) 1)
Multipersistence group H[(1 2),(2 2)]_{1}
Component Z
Component Z
> (multiprst-gnrts K '(1 2) '(2 2) 1)
({CMBN 1}<1 * AB><-1 * AC><1 * BC>
```

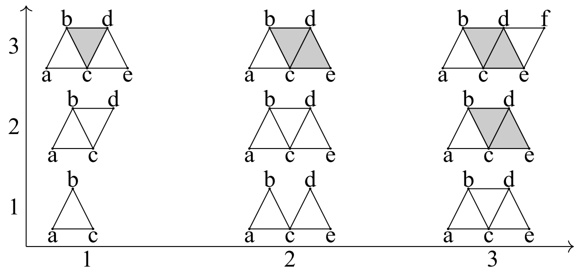

Fig. 2. Second filtration for the small simplicial complex filtered over $\mathbb{Z}^2$.

```
 {CMBN 1}<-1 * AB><1 * AC><-1 * BD><1 * CD>)
> (multiprst-group K '(1 2) '(3 3) 1)
Multipersistence group H[(1 2),(3 3)]_{1}
Component Z
> (multiprst-gnrts K '(1 2) '(3 3) 1)
({CMBN 1}<1 * AB><-1 * AC><1 * BC>)
```

In this case, we can observe that all the persistent homology groups are free; in Section 6 we will present meaningful examples of results with non-null torsion coefficients.

Let us finish this section by observing with a simple example that the rank invariant is not complete, in the sense that sometimes it is unable to discriminate between *different* filtrations (yielding non-isomorphic persistence modules). To this aim, let us consider a second filtration for the example of Fig. 1, given by Fig. 2. The persistence modules in 1-homology associated with the two filtrations are not isomorphic, but the rank invariant of both filtrations is the same, as one can easily verify.

5. A descriptor for birth-death of homology classes and a new invariant

Consider the case of 1-parameter persistent homology, defined from $\mathbb{Z}$-filtrations. We recall the definition

$$M_n^{i,j} := \frac{F_iC_n \cap d(F_jC_{n+1}) + F_{i-1}C_n}{F_iC_n \cap d(F_{j-1}C_{n+1}) + F_{i-1}C_n} \cong \frac{F_iC_n \cap d(F_jC_{n+1})}{F_iC_n \cap d(F_{j-1}C_{n+1}) + d(F_jC_{n+1}) \cap F_{i-1}C_n} \tag{14}$$

of *birth-death modules* given in Romero et al. (2014), therein denoted $BD_n^{i,j}$. When homology is computed over a field, the rank of $M_n^{i,j}$ is given by the quantity $\mu_n^{i,j}$ of equation (1), representing the number of homology classes which are born at step i (meaning that these classes are present at step i but they are not present at the previous step $i-1$) and die at step j of the filtration (meaning that they are present at the previous step $j-1$ but they are not present at step j because they are boundaries or they merge with another class).

For multipersistence, the concepts of birth and death of a homology class cannot be immediately generalized from the 1-parameter case. For example, in Fig. 1 we cannot say that the 1-homology class (1-hole) corresponding to the generator $1*bc - 1*bd + 1*cd$ is born at a unique particular position of $\mathbb{Z}^2$ (because it is present at both positions $(1,2)$ and $(2,1)$ and for both of them it is not present at a previous step). As we have seen, this issue can be solved by considering one-critical filtrations (Section 2), which arise quite naturally in some applications. A more serious problem is that, because of the lack of a decomposition theorem for multipersistence modules like the one for single-parameter persistence, which is a consequence of the complexity of indecomposables for multipersistence modules (see Section 2.1, Buchet and Escolar (2018), Dey and Xin (2019)), the definition of birth and death can depend on the choice of bases for each $H_n(F_v)$.

Interesting approaches to extend the ideas of birth and death of homology classes to multipersistence are proposed in the papers Harrington et al. (2019) and Miller (2017). For the scope of this work, the most relevant approach is described in the PhD thesis Thomas (2019): through the notion of *multirank invariant*, the author establishes a rigorous way for counting births and deaths across a generalized filtration. Nonetheless, an additional hypothesis on the positions to compare in the filtration is needed in order for the count of births and deaths to agree with the intuitive idea, like in the case of 1-parameter persistence. In what follows we choose a more "empirical" approach, mimicking the formula (14) to define a descriptor which, as we will see in some examples, is able to extract birth-death information from a given multiparameter filtration that agrees with the intuition of birth and death of homology classes. The computation of the descriptor is obtained via a modification of our Kenzo programs computing spectral systems. In this section, we also introduce a new invariant, defined using the spectral system associated with a filtration, and observe that it has some similarities with the birth-death descriptor. In order to define these new notions we consider, along with $\mathbb{Z}^m$, the poset of *downsets* of $\mathbb{Z}^m$, which is used in Matschke (2013) to gain more options in the construction of generalized spectral sequences and which will allow us to say that a homology class is born or dead at different positions in $\mathbb{Z}^m$.

Definition 15. A *downset* of $\mathbb{Z}^m$ is a subset $A \subseteq \mathbb{Z}^m$ such that if $Q \leq P$ in $\mathbb{Z}^m$ and $P \in A$, then $Q \in A$; the poset $D(\mathbb{Z}^m)$ is the collection of all downsets of $\mathbb{Z}^m$, endowed with the partial order given by inclusion $\subseteq$.

Filtering data with respect to m parameters produces in a natural way, in addition to the $\mathbb{Z}^m$-filtration $(F_P)_{P\in\mathbb{Z}^m}$ we used in the previous sections, also a $D(\mathbb{Z}^m)$-filtration (F_p) defined, for each $p \in D(\mathbb{Z}^m)$, as $F_p := \sum_{P\in p} F_P$. Moreover, we will observe in Section 5.1 that computing the terms of the spectral system over $D(\mathbb{Z}^m)$ produces more topological information than the rank invariant. In particular, (some terms of) the spectral system of the filtrations defined in Figs. 1 and 2 are different; the spectral system of the filtration over $D(\mathbb{Z}^m)$ can therefore be considered as an invariant associated with a filtration which allows to discriminate between a larger number of topological features.

At this point, it seems natural to investigate possible relations between the rank invariant and the spectral system over $D(\mathbb{Z}^m)$, as we did in Section 3 for $\mathbb{Z}^m$-filtrations. In this case, since there is no natural additive structure on $D(\mathbb{Z}^m)$ that turns it into a partially ordered abelian group, we have to be more subtle. The easiest way to construct a partially ordered abelian group starting from $D(\mathbb{Z}^m)$ is to consider the translation of a fixed downset $p \in D(\mathbb{Z}^m)$. Denoting T_p the family of all downsets of $D(\mathbb{Z}^m)$ obtained translating p by any $v \in \mathbb{Z}^m$, we see that $(T_p, \text{translation}, \subseteq)$ is a partially ordered abelian group. We can now apply the results in Section 3, including (10), using the poset $(T_p, \text{translation}, \subseteq)$, and combining this with results on isomorphic terms within a spectral system (Matschke, 2013, Lemma 3.8) one can obtain interesting relations, the study of which is outside the scope of this work.

For the sake of exposition, we organize the rest of this section in two parts, respectively devoted to the descriptor for birth and death of homology classes and to the new invariant defined from the spectral system. Both subsections contain examples of computations with our programs in Kenzo.

5.1. The birth-death descriptor

We introduce the descriptor in the most general framework, starting from a general $\mathbb{Z}^m$-filtration (F_P) and considering $F_p := \sum_{P\in p} F_P$ for any $p \in D(\mathbb{Z}^m)$, even if the most interesting uses are arguably in some particular situations, for example restricting the "shape" of the downsets p or assuming that the $\mathbb{Z}^m$-filtration (F_P) is induced by a one-critical (see Section 2) $\mathbb{Z}^m$-filtration (K_P) of simplicial complexes.

Consider a downset $p \in D(\mathbb{Z}^m)$. We are interested in $D(\mathbb{Z}^m)$-filtrations canonically associated with *finite* $\mathbb{Z}^m$-filtrations, so we are actually working with the poset of downsets $D(\mathbb{Z}^m_{\geq 0})$. A collection $\{P_1, \ldots, P_k\}$ of points of $\mathbb{Z}^m$ is the *minimal set of generators* of p if it is the minimal set such that, for each point of $P \in p$, it holds $P \leq P_j$ for some $j \in \{1, \ldots, k\}$. In this case, we denote $\hat{F}_p C_n := \cap_{j=1}^k F_{P_j} C_n$. Analogously, consider a downset $b \in D(\mathbb{Z}^m)$ and its minimal set of generators $\{B_1, \ldots, B_r\}$. We now give our definition of a descriptor for birth and death of homology classes.

Definition 16. Let (F_P) be a $\mathbb{Z}^m$-filtration and consider the canonically associated $D(\mathbb{Z}^m)$-filtration $(F_p := \sum_{P\in p} F_P)$. For each $p \leq b$ in $D(\mathbb{Z}^m)$ we define

$$M_n^{p,b} := \frac{\hat{F}_n^{p,b}}{A_n^{p,b} + B_n^{p,b}}$$

where the numerator is

$$\hat{F}_n^{p,b} := \hat{F}_p C_n \cap d(\hat{F}_b C_{n+1})$$

and the denominator has summands

$$A_n^{p,b} := \textstyle\sum_Q \hat{F}_n^{p,b} \cap F_Q C_n + \sum_X \hat{F}_n^{p,b} \cap F_X C_n,$$

with the sums respectively ranging over $Q \in \mathbb{Z}^m$ not comparable with the points P_j and $X \in p \setminus \{P_1, \ldots, P_k\}$, and

$$B_n^{p,b} := \textstyle\sum_R \hat{F}_n^{p,b} \cap d(F_R C_{n+1}) + \sum_Y \hat{F}_n^{p,b} \cap d(F_Y C_{n+1}),$$

with the sums respectively ranging over $R \in \mathbb{Z}^m$ not comparable with the points B_j and $Y \in b \setminus \{B_1, \ldots, B_r\}$.

Intuitively, the groups $M_n^{p,b}$ try to capture the homology classes being born in F_p and dying in F_b, where now the downsets p and b may be generated by several points in $\mathbb{Z}^m$ to deal with the complexity of a filtration with m parameters and the fact that a homology class can appear and disappear at different non-comparable positions. As we said before, the notions of birth and death are not rigorous as in the (single-parameter) persistent homology framework because they depend on the choice of bases for each $H_n(F_v)$. To understand the idea behind Definition 16, let us focus at first on the numerator $\hat{F}_n^{p,b} = \hat{F}_p C_n \cap d(\hat{F}_b C_{n+1})$ of $M_n^{p,b}$. The fact that $\hat{F}_p C_n := \cap_{j=1}^k F_{P_j} C_n$ tells that we want to consider n-cycles that have a common representative at all positions $\{P_1, \ldots, P_k\}$, which bound an $(n+1)$-chain of $\hat{F}_b C_{n+1} := \cap_{j=1}^r F_{B_j} C_{n+1}$. Notice that, depending on the application, the dependence of $M_n^{p,b}$ on a common representative may be a relevant issue. At the denominator of $M_n^{p,b}$, the term $A_n^{p,b}$ is given by the elements of $\hat{F}_n^{p,b}$ that appear also in positions non comparable with $P_1, \ldots, P_k$ or "before" these points, while the term $B_n^{p,b}$ is given by the elements of $\hat{F}_n^{p,b}$ that become boundaries also in positions non comparable with $B_1, \ldots, B_r$ or "before" these points.

As we mentioned, Definition 16 becomes particularly interesting in some simpler situations. First of all, observe that if $m = 1$ the descriptor coincides with the module of (14), for all $i \leq j$. To better understand the definition for $m = 2$, let us consider downsets generated by just one point in $\mathbb{Z}^2$, which is equivalent to considering a $\mathbb{Z}^2$-filtration (F_P). Moreover, let us assume that the filtration is finite: recalling Definition 3, this means that the relevant spaces of the filtration have indices in the finite grid with lower-left corner $(0, 0)$ and upper-right corner $w = (w_1, w_2) \in \mathbb{Z}^2$. In this case for each $P = (p_1, p_2) \leq B = (b_1, b_2)$ in $\mathbb{Z}^2$ we have

$$M_n^{P,B} := \frac{F_n^{P,B}}{A_n^{P,B} + B_n^{P,B}}$$

where

$$\begin{aligned}
F_n^{P,B} &:= F_P C_n \cap d(F_B C_{n+1}),\\
A_n^{P,B} &:= F_n^{P,B} \cap F_{(p_1-1,w_2)} C_n + F_n^{P,B} \cap F_{(w_1,p_2-1)} C_n,\\
B_n^{P,B} &:= F_n^{P,B} \cap d(F_{(b_1-1,w_2)} C_{n+1}) + F_n^{P,B} \cap d(F_{(w_1,b_2-1)} C_{n+1}).
\end{aligned}$$

If the filtration (F_P) is induced by a one-critical $\mathbb{Z}^2$-filtration (K_P) of simplicial complexes, the terms $A_n^{P,B}$ and $B_n^{P,B}$ in the above formula can be further simplified by virtue of Fact 5:

$$A_n^{P,B} := F_n^{P,B} \cap F_{(p_1-1,p_2)}C_n + F_n^{P,B} \cap F_{(p_1,p_2-1)}C_n,$$
$$B_n^{P,B} := F_n^{P,B} \cap d(F_{(b_1-1,w_2)}C_{n+1}) + F_n^{P,B} \cap d(F_{(w_1,b_2-1)}C_{n+1}).$$

Notice that, again by Fact 5, for one-critical filtrations the case of downsets generated by just one point in $\mathbb{Z}^2$ covers all possibilities. The example we just presented for $\mathbb{Z}^2$ can be easily generalized to $\mathbb{Z}^m$-filtrations. In Section 5.2 we introduce an invariant that inspired the definition of $M_n^{p,b}$.

Using again our previous programs for computing spectral systems, we have implemented in Kenzo functions for computing the groups $M_n^{p,b}$ which, as before, produce not only the groups but also the generators. For example, let us consider again the filtered complex in Fig. 1 and the downsets $p = ((1,2),(2,1))$ (meaning *generated* by $\{(1,2),(2,1)\}$) and $b = ((1,3),(3,2))$. The group $M_1^{p,b}$ is equal to $\mathbb{Z}$, with generator $1 * bc - 1 * bd + 1 * cd$. This means intuitively that the homology class corresponding to the boundary of the triangle bcd is born at positions $(1,2)$ and $(2,1)$ and dies at positions $(1,3)$ and $(3,2)$.

```
> (multiprst-m-group K (list '(1 2) '(2 1))
           (list '(1 3) '(3 2)) 1)
Multipersistence group M[((1 2) (2 1)),((1 3) (3 2))]_{1}
Component Z
> (multiprst-m-gnrts K (list '(1 2) '(2 1))
           (list '(1 3) '(3 2)) 1)
({CMBN 1}<1 * BC><-1 * BD><1 * CD>)
```

One of the advantages of the use of the poset $D(\mathbb{Z}^m)$ and the definition of this new descriptor is that it makes it possible to distinguish filtrations which, as we have seen in Section 4, sometimes have the same rank invariant. Let us consider again the generalized filtrations described in Figs. 1 and 2 (with the same rank invariant) and the downsets $p = ((1,3),(2,2),(3,1))$ and $b = ((2,3),(3,2))$; the group $M_1^{p,b}$ is equal to $\mathbb{Z}$ in the first filtration, with generator $1 * cd - 1 * ce + 1 * de$ and the 0-group (NIL) in the second one (because in that filtration this homology class is born at a smaller downset, $((1,3),(2,1))$).

```
> (multiprst-m-group K (list '(1 3) '(2 2) '(3 1))
             (list '(2 3) '(3 2)) 1)
Multipersistence group M[((1 3) (2 2) (3 1)),((2 3) (3 2))]_{1}
Component Z
> (multiprst-m-gnrts K (list '(1 3) '(2 2) '(3 1))
             (list '(2 3) '(3 2)) 1)
({CMBN 1}<1 * CD><-1 * CE><1 * DE>)
> (multiprst-m-group K2 (list '(1 3) '(2 2) '(3 1))
             (list '(2 3) '(3 2)) 1)
Multipersistence group M[((1 3) (2 2) (3 1)),((2 3) (3 2))]_{1}
NIL
```

5.2. A new invariant defined from the spectral system

In this subsection we focus on the behavior of the differential in a spectral system and use it to define an invariant for persistence over a poset I. We start by studying the case of classical spectral sequences (associated with $\mathbb{Z}$-filtrations), a situation where our new invariant defined as the image of differentials coincides with Definition 16 and (14). We then extend our definition to filtrations indexed over general posets, proving that it yields an invariant for generalized persistence.

Given a $\mathbb{Z}$-filtration (F_p), consider two 4-tuples of indices as follows:

$$\begin{array}{ccccccccccc} & & & & p & \le & p+r-1 & \le & p+r & \le & p+2r-1 \\ & & & & \| & & \| & & & & \\ p-r & \le & p-1 & \le & p & \le & p+r-1 & & & & \end{array}$$

The differential $(d^r_{p+r})_{n+1} = d^r_{p+r,q-r+1} : E^r_{p+r,q-r+1} \to E^r_{p,q}$ of the rth page of the spectral sequence can be written in the following equivalent way (see Lemma 10):

$$(d^r_{p+r})_{n+1} : S_{n+1}[p, p+r-1, p+r, p+2r-1] \to S_n[p-r, p-1, p, p+r-1].$$

Let us consider the image of this differential:

$$\mathcal{I}_n^{p,p+r} := \operatorname{Im}(d^r_{p+r})_{n+1} = \frac{F_pC_n \cap d(F_{p+r}C_{n+1})}{F_pC_n \cap d(F_{p+r-1}C_{n+1}) + F_{p-1}C_n \cap d(F_{p+r}C_{n+1})}. \tag{15}$$

Working over a fixed field $\mathbb{F}$ we have the following result:

Proposition 17. *Given a $\mathbb{Z}$-filtration (F_p), knowledge of $\{\dim_{\mathbb{F}} \mathcal{I}_n^{p,p+r}\}$ and the dimensions of the 0-page of the spectral sequence $\{\dim_{\mathbb{F}} E^0_{p,q}\}$ is equivalent to knowledge of the dimensions of all the terms of the spectral sequence $\{\dim_{\mathbb{F}} E^r_{p,q}\}$.*

Proof. For $r = 0$, consider the chain complexes

$$\cdots \to E^0_{p,q+1} \xrightarrow{d^0_{p,q+1}} E^0_{p,q} \xrightarrow{d^0_{p,q}} E^0_{p,q-1} \to \cdots$$

and observe that, for example, $\dim_{\mathbb{F}} E^0_{p,q+1} = \dim_{\mathbb{F}} \operatorname{Ker} d^0_{p,q+1} + \dim_{\mathbb{F}} \operatorname{Im} d^0_{p,q+1}$ and, since the 1-page is obtained by taking homology, $\dim_{\mathbb{F}} E^1_{p,q} = \dim_{\mathbb{F}} \operatorname{Ker} d^0_{p,q} + \dim_{\mathbb{F}} \operatorname{Im} d^0_{p,q+1}$. Therefore, knowing $\dim_{\mathbb{F}} \operatorname{Im} d^0_{p,q}$ and $\dim_{\mathbb{F}} E^0_{p,q}$ for all p and q allows to determine $\dim_{\mathbb{F}} \operatorname{Ker} d^0_{p,q}$ and thus $\dim_{\mathbb{F}} E^1_{p,q}$, for all p and q. The result is obtained by iterating this argument. $\square$

As explained in Basu and Parida (2017) and Section 3, for $\mathbb{Z}$-filtrations the collection $\{\dim_{\mathbb{F}} E^r_{p,q}\}$ is an invariant equivalent to persistent Betti numbers $\{\beta_n^{s,t}\}$. Proposition 17 tells therefore that also $\{\dim_{\mathbb{F}} \mathcal{I}_n^{p,p+r}\}$, together with $\{\dim_{\mathbb{F}} E^0_{p,q}\}$, is an invariant equivalent to persistent Betti numbers. Notice that, since $E^0_{p,q} = F_pC_n/F_{p-1}C_n$, the collection $\{\dim_{\mathbb{F}} E^0_{p,q}\}$ contains information equivalent to $\{\dim_{\mathbb{F}} F_pC_n\}$.

We introduce now the natural generalization of $\mathcal{I}_n^{p,p+r}$ for filtrations indexed over a general poset I. Let (F_i) be an I-filtration, and consider two 4-tuples of indices as follows:

$$\begin{array}{ccccccccccc} & & z_2 & \le & s_2 & \le & p_2 & \le & b_2 \\ & & \| & & \| & & & & \\ z_1 & \le & s_1 & \le & p_1 & \le & b_1 & & \end{array}$$

As before, by Lemma 10 there is a differential

$$d_{n+1} : S_{n+1}[z_2, s_2, p_2, b_2] \to S_n[z_1, s_1, p_1, b_1].$$

We imitate (15) and define

$$\mathcal{I}_n[s_1, p_1, s_2, p_2] := \operatorname{Im} d_{n+1} = \frac{F_{p_1}C_n \cap d(F_{p_2}C_{n+1})}{F_{p_1}C_n \cap d(F_{s_2}C_{n+1}) + F_{s_1}C_n \cap d(F_{p_2}C_{n+1})}. \tag{16}$$

Notice that the indices z_1 and b_2 do not influence the expression (16), which depends only on the indices $s_1 \le p_1 \le s_2 \le p_2$.

In order to show that $\{\mathcal{I}_n[s_1, p_1, s_2, p_2]\}$ is an invariant, let us review some facts on morphisms of I-filtered chain complexes, which are defined generalizing in the natural way the notion of morphisms of $\mathbb{Z}$-filtered chain complexes:

Definition 18. Let (C_*, F) and (C'_*, F') be two I-filtered chain complexes, respectively endowed with filtrations $F = (F_i)_{i\in I}$ and $F' = (F'_i)_{i\in I}$. A *morphism of I-filtered chain complexes* $f : (C_*, F) \to (C'_*, F')$ is a chain map $f : C_* \to C'_*$ compatible with the filtrations, that is satisfying

$$f(F_iC_*) \subseteq F'_iC'_*,$$

for all $i \in I$.

Denote by $(S_n[z,s,p,b])$ and $(S'_n[z,s,p,b])$ the spectral systems associated respectively with the I-filtered chain complexes (C_*, F) and (C'_*, F'). A morphism of I-filtered chain complexes $f : (C_*, F) \to (C'_*, F')$ induces, for any 4-tuple of indices $z \le s \le p \le b$ in I, morphisms

$$f_n^{z,s,p,b} : S_n[z,s,p,b] \to S'_n[z,s,p,b]$$

that commute with the differentials of the spectral systems. The construction of the spectral system associated with an I-filtered chain complex is *functorial*, meaning that for each 4-tuple of indices $z \le s \le p \le b$ it holds $\mathrm{Id}^{z,s,p,b} = \mathrm{Id}_{S[z,s,p,b]}$ and, for each pair of morphisms f, g of I-filtered chain complexes such that the composition gf is defined, it holds $(gf)^{z,s,p,b} = g^{z,s,p,b} f^{z,s,p,b}$. Recalling Fact 7, the last equality appears evident from the following commutative diagram:

$$\begin{array}{ccccc}
F_p \cap d^{-1}(F_z) & \xrightarrow{f} & F'_p \cap d^{-1}(F'_z) & \xrightarrow{g} & F''_p \cap d^{-1}(F''_z) \\
\downarrow q & & \downarrow q' & & \downarrow q'' \\
S_n[z,s,p,b] & \xrightarrow{f_n^{z,s,p,b}} & S'_n[z,s,p,b] & \xrightarrow{g_n^{z,s,p,b}} & S''_n[z,s,p,b].
\end{array}$$

We now prove that $\{\mathcal{I}_n[s_1, p_1, s_2, p_2]\}$ is an invariant for generalized persistence over a poset I. Notice that we use a stronger hypothesis than just isomorphism of persistence modules.

Proposition 19. *If a morphism of I-filtered chain complexes $f : (C_*, F) \to (C'_*, F')$ induces isomorphisms of persistence modules, then for each $z \le s \le p \le b$ in I it induces isomorphisms*

$$f_n^{z,s,p,b} : S_n[z,s,p,b] \to S'_n[z,s,p,b].$$

Proof. We use here the spectral system notation $S_n[-\infty, -\infty, p, p] = H_n(F_p)$ for homology, for each $p \in I$ (see Section 2). By equation (11) in Section 3, if $p \le u$ and $b \le t$ in I then

$$S_n[-\infty, -\infty, p, t] = \mathrm{Im}(\ell : S_n[-\infty, -\infty, p, b] \to S_n[-\infty, -\infty, u, t]),$$

where ℓ denotes every map induced by inclusion, and in particular $S_n[-\infty, -\infty, p, t] = \mathrm{Im}(\ell : H_n(F_p) \to H_n(F_t))$. Since by hypothesis $f : (C_*, F) \to (C'_*, F')$ induces isomorphisms of persistence modules, for each $p \le t$ the map induced by f

$$f_n^{-\infty,-\infty,p,t} : S_n[-\infty, -\infty, p, t] \to S'_n[-\infty, -\infty, p, t] \tag{17}$$

is an isomorphism. Considering long exact sequences like (10) of Section 3 we have a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\ell} & S_n[-\infty,-\infty,p,b] & \xrightarrow{\ell} & S_n[z,s,p,b] & \xrightarrow{k} & S_{n-1}[-\infty,-\infty,z,s] & \xrightarrow{\ell} & \cdots \\
\downarrow f & & \downarrow f & & \downarrow f_n^{z,s,p,b} & & \downarrow f & & \downarrow f \\
\cdots & \xrightarrow{\ell'} & S'_n[-\infty,-\infty,p,b] & \xrightarrow{\ell'} & S'_n[z,s,p,b] & \xrightarrow{k'} & S'_{n-1}[-\infty,-\infty,z,s] & \xrightarrow{\ell'} & \cdots
\end{array}$$

for each $z \le s \le p \le b$, where the vertical maps denoted by f (for simplicity) are isomorphisms like (17), and $f_n^{z,s,p,b} : S_n[z,s,p,b] \to S'_n[z,s,p,b]$ is the map induced by $f : (C_*, F) \to (C'_*, F')$. By the Five Lemma (MacLane, 1963, Lemma 3.3), we can conclude that $f_n^{z,s,p,b}$ is an isomorphism. □

Corollary 20. *In the situation of Proposition 19 it holds that $\mathcal{I}_n[s_1, p_1, s_2, p_2] \cong \mathcal{I}'_n[s_1, p_1, s_2, p_2]$ for all $s_1 \le p_1 \le s_2 \le p_2$ and all n.*

Proof. The claim follows from the commutativity of

$$\begin{array}{ccc} S_{n+1}[z_2,s_2,p_2,b_2] & \xrightarrow{d_{n+1}} & S_n[z_1,s_1,p_1,b_1] \\ {\scriptstyle f_{n+1}^{z_2,s_2,p_2,b_2}}\downarrow & & \downarrow{\scriptstyle f_n^{z_1,s_1,p_1,b_1}} \\ S'_{n+1}[z_2,s_2,p_2,b_2] & \xrightarrow{d'_{n+1}} & S'_n[z_1,s_1,p_1,b_1] \end{array}$$

where by Proposition 19 the vertical maps are isomorphisms. □

As done for the descriptor $M_n^{p,b}$, we have also implemented the construction of the new invariant $\mathcal{I}_n[s_1,p_1,s_2,p_2]$ using our previous programs for computing spectral systems, producing not only the groups but also the generators. We consider again the examples of filtered complexes in Figs. 1 and 2 and show that the new invariant is able to distinguish between both filtrations.

```
> (multiprst-i-group K (list '(1 1) ) (list '(1 2) '(2 1))
(list '(1 2) '(2 1)) (list '(1 3) '(3 2)) 1)
Multipersistence group I[((1 1)),((1 2) (2 1)),((1 2) (2 1)),
((1 3) (3 2))]_{1}
Component Z
> (multiprst-i-group K2 (list '(1 1) ) (list '(1 2) '(2 1))
(list '(1 2) '(2 1)) (list '(1 3) '(3 2)) 1)
Multipersistence group I[((1 1)),((1 2) (2 1)),((1 2) (2 1)),
((1 3) (3 2))]_{1}
Component Z
Component Z
```

It is clear that, for $D(\mathbb{Z}^m)$-filtrations, computing $\mathcal{I}_n[s_1,p_1,s_2,p_2]$ for *any* choice of elements $s_1 \le p_1 \le s_2 \le p_2$ in $D(\mathbb{Z}^m)$ determines a combinatorial explosion and is usually unfeasible in practice. We think that a good choice consists in considering p_1 and p_2 as generated (respectively) by single points P and B in $\mathbb{Z}^m$, with $P \le B$, and setting $s_1 := p_1 \setminus \{P\}$ and $s_2 := p_2 \setminus \{B\}$. Incidentally, this choice is also meaningful in terms of connections within the spectral system (see (Matschke, 2013, Sect. 3)). In this setting, we can notice that the definition of $\mathcal{I}_n[s_1,p_1,s_2,p_2]$ is similar to the descriptor $M_n^{p,b}$ introduced above, especially in the case of downsets generated by single points.

6. Effective homology for infinitely generated spaces

Effective homology (Rubio and Sergeraert (2002, 2006)) is a technique developed to computationally determine the homology of complicated spaces. We briefly introduce the notions necessary to understand the method, before showing how it can be used in the context of persistent homology.

Definition 21. A *reduction* $\rho := (D_* \Rightarrow C_*)$ between two chain complexes D_* and C_* is a triple (f,g,h) where: (a) The components f and g are chain complex morphisms $f: D_* \to C_*$ and $g: C_* \to D_*$; (b) The component h is a morphism of graded modules $h: D_* \to D_{*+1}$ of degree $+1$; (c) The following relations must be satisfied: (1) $fg = \mathrm{id}_{C_*}$; (2) $gf + d_{D_*}h + hd_{D_*} = \mathrm{id}_{D_*}$; (3) $fh = 0$; (4) $hg = 0$; (5) $hh = 0$.

Since f is a chain equivalence between D_* and C_*, in particular the homology groups $H_n(D_*)$ and $H_n(C_*)$ are canonically isomorphic, for each n.

Definition 22. An *effective* chain complex C_* is a free chain complex (i.e., a chain complex consisting of free $\mathbb{Z}$-modules) where each group C_n is finitely generated, and there is an algorithm that returns a $\mathbb{Z}$-base in each degree n.

Intuitively, an effective chain complex C_* is a chain complex whose homology can be easily determined via standard diagonalization algorithms (see Kaczynski et al. (2004)).

Definition 23. A chain complex C_* has *effective homology* if there exist a chain complex D_*, an *effective* chain complex EC_* and two reductions $C_* \Leftarrow D_* \Rightarrow EC_*$.

The technique of effective homology has been implemented in the system Kenzo, which is able to automatically construct the reductions $C_* \Leftarrow D_* \Rightarrow EC_*$ in several situations arising in algebraic topology and homological algebra. In the scenario of the previous definition, the method of effective homology allows to determine the homology groups of the original chain complex C_* by using EC_* to perform the computations. In this way, Kenzo is able to determine homology and homotopy groups of complicated spaces, even when the chain complex C_* is not finitely generated (resulting thus untreatable by standard algorithms), and has shown its potentiality successfully computing previously unknown results (Rubio and Sergeraert (2006)).

Now, we want to show how the effective homology technique can be applied to compute persistent homology groups. First, let us study the behavior of reductions when we introduce I-filtrations on the involved chain complexes. Let F (resp. F') be an I-filtration of a chain complex D_* (resp. C_*), and let S (resp. S') denote the terms of the associated spectral system. In Guidolin and Romero (2018) we stated the following result.

Theorem 24. *(Guidolin (2018)) Let $\rho = (f, g, h) : D_* \Rightarrow C_*$ be a reduction between the I-filtered chain complexes (D_*, F) and (C_*, F'), and suppose that f and g are compatible with the filtrations, that is, $f(F_i) \subseteq F'_i$ and $g(F'_i) \subseteq F_i$ for all $i \in I$. Then, given four indices $z \le s \le p \le b$ in I, the map f induces for each n an isomorphism*

$$f_n^{z,s,p,b} : S_n[z,s,p,b] \to S'_n[z,s,p,b]$$

whenever the homotopy $h : D_* \to D_{*+1}$ satisfies the conditions $h(F_z) \subseteq F_s$ and $h(F_p) \subseteq F_b$.

Proof. Remember the following properties of a reduction:

- $fg = \mathrm{Id}_{C_*}$,
- $gf + d_{D_*}h + hd_{D_*} = \mathrm{Id}_{D_*}$.

The first property implies that, for any 4-tuple of indices $z \le s \le p \le b$ in I, we have the induced maps $(fg)_n^{z,s,p,b} = (\mathrm{Id}_{C_*})_n^{z,s,p,b}$ between terms of the spectral system. Then, by functoriality,

$$f_n^{z,s,p,b} g_n^{z,s,p,b} = \mathrm{Id}_{S'_n[z,s,p,b]} .$$

The second property means that h is a chain homotopy between gf and Id_{D_*}. Then, a generalization of (MacLane, 1963, Prop. 3.5) whose details are worked out in Guidolin (2018) yields induced maps $(gf)_n^{z,s,p,b} = (\mathrm{Id}_{D_*})_n^{z,s,p,b}$ whenever h satisfies the conditions $h(F_z) \subseteq F_s$ and $h(F_p) \subseteq F_b$. Therefore, again by functoriality,

$$g_n^{z,s,p,b} f_n^{z,s,p,b} = \mathrm{Id}_{S_n[z,s,p,b]}$$

whenever h satisfies $h(F_z) \subseteq F_s$ and $h(F_p) \subseteq F_b$. $\square$

This result is very useful, and is used also in Guidolin and Romero (2020) to study how the effective homology technique can be leveraged to compute the Serre spectral system, a generalization of the classical Serre spectral sequence.

Now, taking into account the relations between multipersistence and spectral systems studied in Section 3, we obtain the following corollary.

Corollary 25. *In the situation of Theorem 24, we have in particular that the map f induces isomorphisms*

$$H_n^{p,b}(D_*) = S_n[-\infty, -\infty, p, b] \to H_n^{p,b}(C_*) = S'_n[-\infty, -\infty, p, b]$$

whenever the homotopy $h : D_* \to D_{*+1}$ satisfies the condition $h(F_p) \subseteq F_b$.

Clearly, if in Corollary 25 the map h is also compatible with the filtrations, we can conclude that $H_n^{p,b}(D_*) \cong H_n^{p,b}(C_*)$ for all $p \leq b$ in I. This new result provides us a method for computing persistent homology groups of chain complexes of infinite type when the effective homology (Definition 23) of the chain complex is known. This method has been implemented in Kenzo and can be applied to complicated spaces filtered over general posets. As we mentioned before, it represents a unique feature among the available programs for computing invariants for (generalized) persistent homology.

We prove a result similar to Theorem 24 also for the descriptor $M_n^{p,b}$ introduced in Section 5.2.

Theorem 26. *Let $\rho = (f, g, h) : D_* \Rightarrow C_*$ be a reduction between the I-filtered chain complexes (D_*, F) and (C_*, F'), let $p \leq b$ in I and let us suppose that f and g are compatible with the filtrations and h satisfies the condition $h(\hat{F}^{p,b}) \subseteq \sum_R F_R + \sum_Y F_Y$ with $R \in \mathbb{Z}^m$ not comparable with the points B_j defining the downset b and $Y \in b \setminus \{B_1, \ldots, B_r\}$. Then, the map f induces for each n an isomorphism*

$$f_n^{p,b} : M_n^{p,b}(D_*) \to M_n^{p,b}(C_*).$$

Proof. Remember the formula

$$M_n^{p,b} := \frac{\hat{F}_n^{p,b}}{A_n^{p,b} + B_n^{p,b}}$$

in Definition 16 and the following properties of a reduction:

- $fg = \mathrm{Id}_{C_*}$,
- $gf + d_{D_*}h + hd_{D_*} = \mathrm{Id}_{D_*}$.

The first property implies that, for any pair of indices $p \leq b$ in I, we have the induced maps $(fg)_n^{p,b} = (\mathrm{Id}_{C_*})_n^{p,b}$. Then,

$$f_n^{p,b} g_n^{p,b} = \mathrm{Id}_{M_n^{p,b}}.$$

The second property implies that, given $\sigma \in \hat{F}_n^{p,b}$, we have $\mathrm{Id}(\sigma) = gf(\sigma) + hd(\sigma) + dh(\sigma)$. On the one hand, $hd(\sigma) = 0$ (since $\sigma \in \hat{F}_n^{p,b}$ and $dd = 0$); on the other hand, we know that $h(\hat{F}_n^{p,b}) \subseteq \sum_R F_R + \sum_Y F_Y$ and d is compatible with the filtration, so $dh(\sigma) \in B_n^{p,b}$ which is part of the quotient defining $M_n^{p,b}$. Considering now the induced maps on the corresponding quotients, we have the desired expression

$$\mathrm{Id}_{M_n^{p,b}} = g_n^{p,b} f_n^{p,b}. \quad \square$$

Observe that, if the map h is compatible with the filtrations, then we can deduce $M_n^{p,b}(D_*) \cong M_n^{p,b}(C_*)$ for every $p < b$. We conclude this section with a result describing the behavior of reductions (and effective homology) on the invariant $\mathcal{I}$ we introduced in Section 5.2.

Theorem 27. *Let $\rho = (f, g, h) : D_* \Rightarrow C_*$ be a reduction between the I-filtered chain complexes (D_*, F) and (C_*, F'), and suppose that f and g are compatible with the filtrations. Then, given four indices $s_1 \leq p_1 \leq s_2 \leq p_2$ in I, the map f induces for each n an isomorphism*

$$f_n^{s_1,p_1,s_2,p_2} : \mathcal{I}_n[s_1,p_1,s_2,p_2] \to \mathcal{I}'_n[s_1,p_1,s_2,p_2]$$

whenever the homotopy $h : D_* \to D_{*+1}$ *satisfies the condition* $h(F_{p_1}) \subseteq F_{s_2}$.

Proof. We can apply a similar argument to the proof of Corollary 20. In that proof the indices were elements in I such that

$$\begin{array}{ccccccccccc} & & & & z_2 & \leq & s_2 & \leq & p_2 & \leq & b_2 \\ & & & & \| & & \| & & & & \\ z_1 & \leq & s_1 & \leq & p_1 & \leq & b_1 & & & & \end{array}$$

but now we can disregard z_1 and b_2, since they do not intervene in the definition of $\mathcal{I}_n[s_1,p_1,s_2,p_2]$. In other words, we can set "artificially" $z_1 = -\infty$ and $b_2 = \infty$, with $F_{-\infty} := 0$ and $F_\infty := D_*$, and similarly with F' and C_*. Then, the vertical maps in the proof of Corollary 20 are isomorphisms by Theorem 24, for which we only need to assume $h(F_{p_1}) \subseteq F_{s_2}$. □

If in Theorem 27 also the map h is compatible with the filtrations, we clearly have $\mathcal{I}_n[s_1,p_1,s_2,p_2] \cong \mathcal{I}'_n[s_1,p_1,s_2,p_2]$ for every choice of indices $s_1 \leq p_1 \leq s_2 \leq p_2$ in I.

7. Improving the algorithms via discrete vector fields

The ability of the system Kenzo to exploit the relationship between different chain complexes is brought one step further by the use of *discrete vector fields*, a notion introduced by Robin Forman (Forman (1998)) which proved itself incredibly useful in computational algebraic topology. For example, in Mischaikow and Nanda (2013) discrete vector fields are applied to the computation of persistent homology. In what follows we briefly describe how discrete vector fields can simplify the computation of generalized persistent homology in our setting, which possibly involves chain complexes of infinite type. For other applications of discrete vector fields to multipersistence we refer the reader to Scaramuccia et al. (2020) and Landi and Scaramuccia (2019).

Let $C_* = (C_n, d_n)$ be a free chain complex with distinguished $\mathbb{Z}$-bases $B_n \subseteq C_n$, whose elements we call *n-cells*. We will use the notation $C_* = (C_n, d_n, B_n)$ when we want to stress that the chain complex C_* is equipped with distinguished bases.

Definition 28. A *discrete vector field* V on C_* is a collection of pairs of cells $V = \{(\sigma_k; \tau_k)\}_{k \in K}$ satisfying specific conditions (see Romero and Sergeraert, 2010, Definition 5):

- Every component $\sigma_k \in B_n$ is a *regular face* of the corresponding $\tau_k \in B_{n+1}$ (that is, the coefficient of σ_k in $d\tau_k$ is $+1$ or -1).
- Each generator (cell) of C_* appears at most one time in V.

Let us point out that we do not require the distinguished bases B_n or the vector field V to be finite. Observe that our definition is quite general, and does not require the chain complex C_* to be canonically associated with a topological or combinatorial object such as a simplicial complex or a simplicial set; on the other hand, starting from a simplicial complex or a simplicial set K, there are "obvious" distinguished bases B_n for $C_* = C_*(K)$, given by the sets of n-simplices of K.

Definition 29. A pair $(\sigma_j; \tau_j)$ of V is called a *vector*; we use the notations $\tau_j = V(\sigma_j)$ or $\sigma_j = V^{-1}(\tau_j)$ to express the fact that σ_j and τ_j are the components of a vector of V. The cells σ_j and τ_j are called respectively a *source cell* and a *target cell*. A cell $\sigma \in B_n$ which does not appear in the discrete vector field V is called a *critical cell*.

Definition 30. Given a discrete vector field V, a *V-path π of degree n and length m* is a sequence $\pi = \{(\sigma_{j_k}; \tau_{j_k})\}_{0 \leq k < m}$ such that:

- Every pair $(\sigma_{j_k}; \tau_{j_k})$ is a vector of V and τ_{j_k} is an n-cell.

- For every $0 < k < m$, the component σ_{j_k} is a *face* of $\tau_{j_{k-1}}$ (meaning that the coefficient of σ_{j_k} in $d\tau_{j_{k-1}}$ is non-null), non necessarily regular but different from $\sigma_{j_{k-1}}$.

Definition 31. A discrete vector field V is called *admissible* if, for every $n \in \mathbb{Z}$, a function $\lambda_n : B_n \to \mathbb{N}$ is provided such that the length of every V-path starting from $\sigma \in B_n$ is bounded by $\lambda_n(\sigma)$.

The following result, due to Forman (Forman, 1998, § 8), has been generalized in Romero and Sergeraert (2010) to the case of chain complexes not necessarily of finite type.

Theorem 32. *(Forman (1998); Romero and Sergeraert (2010)) Let $C_* = (C_n, d_n, B_n)$ be a free chain complex and $V = \{(\sigma_k; \tau_k)\}_{k \in K}$ be an admissible discrete vector field on C_*. Then the vector field V defines a canonical reduction $\rho = (f, g, h) : (C_n, d_n) \Rightarrow (C_n^c, d'_n)$ where C_n^c is the free $\mathbb{Z}$-module generated by critical n-cells and d'_n is an appropriate differential canonically defined from C_* and V.*

Theorem 32, together with Kenzo's algorithms for automatically constructing admissible discrete vector fields (Romero and Sergeraert (2010)), allows to compute the homology groups $H_n(C_*) \cong H_n(C_*^c)$ working with the chain complex C_*^c of reduced size. We sketch the proof given in Romero and Sergeraert (2010), as we will refer to it for proving Theorem 37.

Proof. For each basis B_n, consider the partition $B_n^t \cup B_n^s \cup B_n^c$ into target, source and critical cells, which induces a decomposition (as $\mathbb{Z}$-modules) of the chain groups: $C_n = C_n^t \oplus C_n^s \oplus C_n^c$. By virtue of this decomposition, each differential d_n can be represented as a 3×3 matrix

$$d_n = \begin{bmatrix} d_{n,1,1} & d_{n,1,2} & d_{n,1,3} \\ d_{n,2,1} & d_{n,2,2} & d_{n,2,3} \\ d_{n,3,1} & d_{n,3,2} & d_{n,3,3} \end{bmatrix}.$$

It can be proven that $d_{n,2,1} : C_n^t \to C_{n-1}^s$ is an isomorphism, and that its inverse $d_{n,2,1}^{-1} : C_{n-1}^s \to C_n^t$ can be made explicit via the recursive formula

$$d_{n,2,1}^{-1}(\sigma) = \varepsilon(\sigma, V(\sigma)) \left(V(\sigma) - \sum_{\sigma' \in B_{n-1}^s \setminus \{\sigma\}} \varepsilon(\sigma', V(\sigma)) d_{n,2,1}^{-1}(\sigma') \right), \tag{18}$$

where $\varepsilon(\sigma, \tau)$ denotes the coefficient of σ in the differential $d\tau$. Then, the differential d' and the maps f, g, h of the reduction can be explicitly defined as follows:

$$\begin{aligned} d'_n &= d_{n,3,3} - d_{n,3,1} d_{n,2,1}^{-1} d_{n,2,3} & f_{n-1} &= \begin{bmatrix} 0 & -d_{n,3,1} d_{n,2,1}^{-1} & 1 \end{bmatrix} \\ g_n &= \begin{bmatrix} -d_{n,2,1}^{-1} d_{n,2,3} \\ 0 \\ 1 \end{bmatrix} & h_{n-1} &= \begin{bmatrix} 0 & d_{n,2,1}^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{19}$$ □

We now want to add I-filtrations to the picture, in order to show the relevance of discrete vector fields in the computation of persistent homology groups for generalized persistence. We have to require that an additional compatibility condition is satisfied:

Definition 33. An I-filtration $F = (F_i)_{i \in I}$ of $C_* = (C_n, d_n, B_n)$ is an I-filtration F of the chain complex (C_n, d_n) that is compatible with faces: if σ is a face of a cell τ, then $(\tau \in F_i \implies \sigma \in F_i)$, for all $i \in I$.

Remark 34. Since we are considering the general framework of chain complexes (C_n, d_n, B_n) equipped with distinguished bases for the chain groups, compatibility with faces (Definition 33) is not automatically satisfied for every I-filtration of chain subcomplexes of (C_n, d_n). As a counterexample, let C_1 be generated by an element x and let C_0 be generated by two elements a, b, with $d(x) = b - a$. Given the poset $I = \{0 < 1\}$, let F_1 be the full chain complex (with generators x, a, b) and let F_0 be generated by x and $b - a$. Then $x \in F_0$ but (in general) $a \notin F_0$.

Notice that in a canonical reduction $\rho : (C_n, d_n) \Rightarrow (C_n^c, d'_n)$, an I-filtration defined on the chain complex C_* canonically induces an I-filtration on C_*^c.

Definition 35. Let $C_* = (C_n, d_n, B_n)$ be a free chain complex with an I-filtration $F = (F_i)_{i \in I}$ and let $V = \{(\sigma_k; \tau_k)\}_{k \in K}$ be a discrete vector field on C_*. If $(\sigma_k \in F_i \iff \tau_k \in F_i)$ for all $i \in I$ and for all $k \in K$ we say that V is *compatible* with the I-filtration F.

Remark 36. Let $F = (F_i)_{i \in I}$ be an I-filtration of $C_* = (C_n, d_n, B_n)$. By Definition 33, if σ is a face of a cell τ, then $(\tau \in F_i \implies \sigma \in F_i)$, for all $i \in I$. Then, if $V = \{(\sigma_j; \tau_j)\}_{j \in J}$ is a discrete vector field compatible with the filtration, for each V-path $\pi = \{(\sigma_{j_k}; \tau_{j_k})\}_{0 \le k < m}$ we can conclude that

$$\sigma_{j_0} \in F_i \implies \tau_{j_m} \in F_i,$$

for each $i \in I$.

Theorem 37. *(Guidolin (2018)) If $C_* = (C_n, d_n, B_n)$ is endowed with an I-filtration $F = (F_i)_{i \in I}$ and $V = \{(\sigma_k; \tau_k)\}_{k \in K}$ is an admissible discrete vector field on C_* which is compatible with F, then the three maps of the canonical reduction $\rho = (f, g, h) : (C_n, d_n) \Rightarrow (C_n^c, d'_n)$ described in Theorem 32 are compatible with the filtrations.*

Proof. We refer to the proof of Theorem 32. Recall the decomposition $C_n = C_n^t \oplus C_n^s \oplus C_n^c$ of the chain groups; on the groups C_n^t, C_n^s and C_n^c consider the "obvious" I-filtrations (of abelian groups) induced by F. Clearly, each component $d_{n,k,\ell}$ (with $k, \ell = 1, 2, 3$) of the differential d is compatible with the filtrations. As the differential d' of C_*^c and the maps f, g, h of the reduction are given by (19), we only need to prove that $d_{n,2,1}^{-1}$ is compatible with the filtrations in order to conclude that d', f, g, h are compatible with the filtrations. For each $\sigma \in C_{n-1}^s$, using the recursive formula (18) we can express $d_{n,2,1}^{-1}(\sigma)$ as a finite sum

$$d_{n,2,1}^{-1}(\sigma) = \sum \lambda_k \tau_{j_k},$$

where the λ_k are coefficients in $\mathbb{Z}$ and each τ_{j_k} is at the end of a V-path starting from σ. Then from Remark 36 follows that $d_{n,2,1}^{-1}$ is compatible with the filtrations. $\square$

Since in particular the map h of $\rho = (f, g, h) : (C_n, d_n) \Rightarrow (C_n^c, d'_n)$ is compatible with the I-filtrations defined on C_* and C_*^c, Corollary 25, Theorem 26 and Theorem 27 tell us that discrete vector fields can be used to improve the computations, with a guarantee that the returned results are correct for *all* persistent homology groups, descriptors $M_n^{p,b}$ and invariants $\mathcal{I}_n[s_1, p_1, s_2, p_2]$:

Corollary 38. *In the situation of Theorem 37, the map f of the reduction $\rho = (f, g, h) : (C_n, d_n) \Rightarrow (C_n^c, d'_n)$ induces isomorphisms:*

- $H_n^{p,b}(C_*) \cong H_n^{p,b}(C_*^c)$ *for all $p \le b$ in I,*
- $M_n^{p,b}(C_*) \cong M_n^{p,b}(C_*^c)$ *for all $p < b$ in I,*
- $\mathcal{I}_n(C_*)[s_1, p_1, s_2, p_2] \cong \mathcal{I}_n(C_*^c)[s_1, p_1, s_2, p_2]$ *for all $s_1 \le p_1 \le s_2 \le p_2$ in I.*

Making use of this result and of Kenzo algorithms for computing admissible discrete vector fields (Romero and Sergeraert (2010)) we have enhanced our programs for the computation of multipersistence.

8. Examples and computations

The algorithms presented in the previous sections have been implemented as a new module for the Kenzo system available at https://github.com/ana-romero/Kenzo-external-modules. In this section we present three different examples of application of our programs.

8.1. Effective example

As a first example showing the functionality of our algorithms, let us consider the following chain complex C_*, which is effective (Definition 22):

$$C_2 = \mathbb{Z}^4 \xrightarrow{d_2} C_1 = \mathbb{Z}^4 \xrightarrow{d_1} C_0 = \mathbb{Z}$$

with differential maps d_2 and d_1 given respectively by the matrices

$$D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix} \qquad \text{and} \qquad D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

Its homology groups are: $H_0(C_*) = 0$, $H_1(C_*) = \mathbb{Z}/4\mathbb{Z}$ and $H_2(C_*) = \mathbb{Z}$. In particular, observe that H_1 is not free.

Let us suppose now that the free groups C_0, C_1 and C_2 are generated respectively by elements $C_0 = \langle a \rangle$, $C_1 = \langle b_1, b_2, b_3, b_4 \rangle$ and $C_2 = \langle c_1, c_2, c_3, c_4 \rangle$. Then, we consider the generalized filtration over $\mathbb{Z}^2$ given by $F_{(1,1)} = \langle a \rangle$, $F_{(1,2)} = \langle a, b_1, b_2, b_3, b_4, c_1, c_2, c_3 \rangle$, $F_{(2,1)} = \langle a, b_3, b_4, c_3 \rangle$ and $F_{(2,2)} = C_*$. We focus on degree of homology $n = 1$ (where the most interesting groups appear) and compute the multipersistence groups, together with their generators, for different indices of the filtration.

```
> (multiprst-group C '(2 2) '(2 2) 1)
Multipersistence group H[(2 2),(2 2)]_{1}
NIL
> (multiprst-group C '(1 2) '(1 2) 1)
Multipersistence group H[(1 2),(1 2)]_{1}
Component Z/2Z
Component Z/4Z
> (multiprst-gnrts C '(1 2) '(1 2) 1)
({CMBN 1}<1 * B3>
 {CMBN 1}<-1 * B2>)
> (multiprst-group C '(2 1) '(2 1) 1)
Multipersistence group H[(2 1),(2 1)]_{1}
Component Z/2Z
Component Z
> (multiprst-gnrts C '(2 1) '(2 1) 1)
({CMBN 1}<1 * B4>
 {CMBN 1}<1 * B3>)
> (multiprst-group C '(2 2) '(2 2) 1)
Multipersistence group H[(2 2),(2 2)]_{1}
Component Z/4Z
> (multiprst-gnrts C '(2 2) '(2 2) 1)
({CMBN 1}<-1 * B2>)
> (multiprst-group C '(1 2) '(2 2) 1)
Multipersistence group H[(1 2),(2 2)]_{1}
Component Z/4Z
> (multiprst-gnrts C '(1 2) '(2 2) 1)
({CMBN 1}<-1 * B2>)
```

```
> (multiprst-group C '(2 1) '(2 2) 1)
Multipersistence group H[(2 1),(2 2)]_{1}
Component Z/2Z
> (multiprst-gnrts C '(2 1) '(2 2) 1)
({CMBN 1}<1 * B4>)
```

From these computations, we can construct the persistence module (for $n = 1$) which can be summarized by the following modules and morphisms:

$$\begin{tikzcd} V_{(1,2)} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \arrow[r, "{[-2\ 0]}"] & V_{(2,2)} = \mathbb{Z}/4\mathbb{Z} \\ V_{(1,1)} = 0 \arrow[u] \arrow[r] & V_{(2,1)} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \arrow[u, "{[0\ 1]}"'] \end{tikzcd}$$

We can also compute our new descriptor of Section 5 and see that the homology class generated by the element b_3 is born at positions $(1, 2)$ and $(2, 1)$ and dies at $(2, 2)$:

```
> (multiprst-m-group C (list '(1 2)) (list '(2 2)) 1)
NIL
> (multiprst-m-group C (list '(2 1)) (list '(2 2)) 1)
NIL
> (multiprst-m-group C (list '(1 2) '(2 1)) (list '(2 2)) 1)
Component Z/2Z
> (multiprst-m-gnrts C (list '(1 2) '(2 1)) (list '(2 2)) 1)
({CMBN 1}<1 * B3>)
```

Moreover, the new invariant $\mathcal{I}_n[s_1, p_1, s_2, p_2]$ can provide useful information on the generalized filtration.

```
> (multiprst-i-group C (list '(1 1)) (list '(1 2)) (list '(1 2))
(list '(2 2)) 1)
Multipersistence group I[((1 1)),((1 2)),((1 2)),((2 2))]_{1}
Component Z/2Z
> (multiprst-i-group C (list '(1 1)) (list '(2 1)) (list '(2 1))
(list '(2 2)) 1)
Multipersistence group I[((1 1)),((2 1)),((2 1)),((2 2))]_{1}
Component Z
```

8.2. Using effective homology

An example of situation where the computation of multipersistence of infinitely generated chain complexes can be relevant involves twisted Cartesian products (May (1967)) of simplicial sets where at least one space is of infinite type. Twisted Cartesian products are obtained as total spaces of *towers of fibrations* (successive fibrations where the total space of each one coincides with the base of the previous one), and multipersistence provides information on the interaction of the homology groups of the different components in the product.

For example, let us consider the first stages of the Whitehead tower for computing the homotopy groups of the sphere S^3, given by the following tower of fibrations:

$$\begin{tikzcd} X_6 \arrow[r] & X_5 \arrow[r] & X_4 \arrow[r] & B = S^3 \\ G_4 = K(\mathbb{Z}_2, 4) \arrow[u] & G_3 = K(\mathbb{Z}_2, 3) \arrow[u] & G_2 = K(\mathbb{Z}, 2) \arrow[u] & \end{tikzcd}$$

The first total space X_4 can be seen as a twisted Cartesian product $X_4 = K(\mathbb{Z}, 2) \times_{\tau_4} S^3$, where $K(\mathbb{Z}, 2)$ is an Eilenberg–MacLane space (May (1967)). The total space X_5 of the second fibration is given by $X_5 = K(\mathbb{Z}_2, 3) \times_{\tau_5} X_4 = K(\mathbb{Z}_2, 3) \times_{\tau_5} (K(\mathbb{Z}, 2) \times_{\tau_4} S^3)$. Finally, the total space X_6 of the third fibration is equal to $X_6 = K(\mathbb{Z}_2, 4) \times_{\tau_6} X_5 = K(\mathbb{Z}_2, 4) \times_{\tau_6} (K(\mathbb{Z}_2, 3) \times_{\tau_5} (K(\mathbb{Z}, 2) \times_{\tau_4} S^3))$. See May (1967) for the construction of this tower, which satisfies $H_n(X_n) \cong \pi_n(S^3)$.

Eilenberg–MacLane spaces $K(\pi, n)$'s are represented in Kenzo by means of the classifying space constructor (see May (1967) for details). In particular, if the group π is not finite (for instance $\mathbb{Z}$), then the set of m-simplices of $K(\pi, n)$ for every $m \geq n$ is infinite and hence $K(\pi, n)$ is of infinite type.

The total space X_6 can be filtered over $D(\mathbb{Z}^3)$ (where $m = 3$ coincides with the number of fibrations) by using the *degeneracy degrees* of the simplices (May (1967)), so that multipersistence can be studied. Let us observe that one of the factors, namely $K(\mathbb{Z}, 2)$, is not of finite type, so the rank invariant can not be directly determined via standard algorithms based on matrix reduction. However, the effective homology method implemented in Kenzo combined with the theoretical guarantee of Corollary 25 makes it possible to determine the multipersistence groups (and their rank).

In this example, our results allow us to reproduce the result $\pi_6(S^3) \cong H_6(X_6) \cong \mathbb{Z}/12\mathbb{Z}$ given by the group $H_6^{((7,7,7)),((7,7,7))}$:

```
> (multiprst-group X6 (list '(7 7 7)) (list '(7 7 7)) 6)
Multipersistence group H[((7 7 7)),((7 7 7))]_{6}
Component Z/12Z
```

In a context like this, the computation of multipersistence can reveal interesting information not only on the homology of individual spaces, but also on the role played by the filtration, as we see for example for the group $H_6^{((6,6,6)),((7,6,6))} = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}$.

```
> (multiprst-group X6 (list '(6 6 6)) (list '(7 6 6)) 6)
Multipersistence group H[((6 6 6)),((7 6 6))]_{6}
Component Z/4Z
Component Z
```

We want to stress the important role of Corollary 25 and Theorem 26 in this situation, as they characterize the persistent homology groups that can be computed correctly using effective homology in terms of the behavior of the homotopy operators h of the involved reductions, which can be determined explicitly.

Notice also that the system Kenzo is able to handle simplicial sets, which are more general and versatile than simplicial complexes; this allows it to deal with a broader variety of situations. The method of effective homology further enlarges the range of objects in algebraic topology it can compute and manipulate. To our knowledge, Kenzo is the only available software to make computations on filtrations of infinitely generated chain complexes like the one we considered in this example.

8.3. *Using discrete vector fields*

As a last example of application of our programs, let us show how discrete vector fields can be used to improve the efficiency when working with chain complexes (or simplicial sets) with a large number of generators. Consider the chain complex associated with the digital image shown in Fig. 3, filtered over $\mathbb{Z}^2$ (more precisely: a 4×4 grid in $\mathbb{Z}^2$). For details and examples on how a digital image yields a simplicial complex and a chain complex we refer the reader to Romero et al. (2016).

In this case the simplicial complex has 203 vertices, 408 edges and 208 triangles. Even if the associated chain complex is not very big, it is convenient to use discrete vector fields to *reduce* it to a smaller one. The paper Guidolin and Romero (2018) includes an algorithm to determine an admissible discrete vector field which is compatible with a given generalized filtration defined on a chain complex of finite type. This discrete vector field can be computed in Kenzo and, when applied to the chain complex of this example, returns an effective chain complex as in Theorem 32 (stored in a slot called `efhm`) with 21 vertices, 23 edges and 5 triangles.

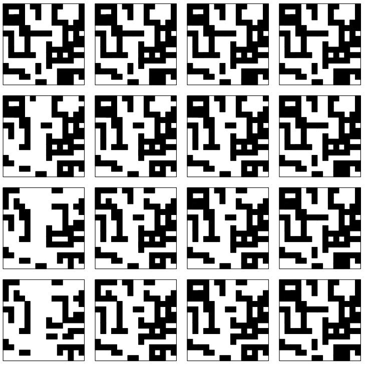

Fig. 3. Digital image filtered over $\mathbb{Z}^2$.

```
> (efhm K3)
[K155 Homotopy-Equivalence K123 <= K123 => K141]
> (setf efK3 (rbcc (efhm K3)))
[K141 Generalized-Filtered-Chain-Complex]
> (length (basis efK3 0))
21
> (length (basis efK3 1))
23
> (length (basis efK3 2))
5
```

In this way, we can significantly improve the computation of the multipersistence groups and our new descriptor $M^{p,b}$ and the invariant with their corresponding generators.

9. Conclusions and further work

We presented a set of programs for performing computations on chain complexes with filtrations defined over posets. The programs allow to compute generalized persistent homology, and in particular some relevant invariants in the context of multipersistence. Although, due to the necessary adjustments to deal with infinite spaces, our programs are not as efficient as previous existing implementations with polynomial complexity, we provide algorithms which are valid in general situations, some of which cannot be tackled by any other method. One fundamental aspect of our implementation consists in the use of the effective homology technique, which makes it possible to handle infinitely generated chain complexes. Another important feature concerns the possibility of defining and using for computation filtrations over general posets. Our programs, improved using discrete vector fields, have been implemented as a new module for the Kenzo system.

We focused our study on filtrations indexed over the posets $\mathbb{Z}^m$ and $D(\mathbb{Z}^m)$, for their relevance in relation with multipersistence. In this respect, a theoretical contribution of our work is the description of the relation between persistent homology and spectral systems in a general scenario, which extends a result valid for persistent homology and spectral sequences arising from $\mathbb{Z}$-filtrations. Furthermore, we introduce a descriptor, which is able to extract birth-death information from multiparameter filtrations, and a new invariant. Both of them have also been implemented in the Kenzo system. We show the connection between both definitions and their discriminative power in the context of multipersistence.

Two fundamental requirements in persistent homology theory are computability and robustness. As a future research direction, we intend to reduce the computational cost for our invariants and to further investigate their behavior with respect to small changes in the multiparameter filtration. As we reviewed in Section 4, several approaches have been proposed to tackle the problems arising with multiparameter filtrations. Since effective homology displays a good behavior with respect to the invariants we considered in this work, studying its applicability to other constructions represents an interesting scope for further research.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that *could have appeared to influence the work reported in* this *paper*.

Acknowledgements

Partially supported by Basque Government, BERC 2018-2021 program; Spanish Ministry of Science, Innovation and Universities, BCAM Severo Ochoa accreditation SEV-2017-0718; Spanish Ministry of Science, Innovation and Universities, project MTM2017-88804-P; University of La Rioja - V Plan Riojano de I+D+I; Italian MIUR Award "Dipartimento di Eccellenza 2018-2022"- CUP: E11G18000350001 and SmartData@PoliTO center for Big Data and Machine Learning technologies. F.V. acknowledges partial support from Intesa Sanpaolo Innovation Center. The funder had no role in study design, data collection, and analysis, decision to publish, or preparation of the manuscript. We thank the anonymous reviewers, including those of the previous conference version of the article, for their helpful comments and suggestions.

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