Fracture Strength: Stress Concentration, Extreme Value Statistics, and the Fate of the Weibull Distribution

Zsolt Bertalan,[1] Ashivni Shekhawat,[2,3] James P. Sethna,[4] and Stefano Zapperi[5,1,*]

[1]*ISI Foundation, Via Alassio 11/c, 10126 Torino, Italy*

[2]*Department of Materials Science and Engineering, University of California, Berkeley, California 94720, USA*

[3]*Materials Science Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*

[4]*LASSP, Physics Department, Cornell University, Ithaca, New York 14853-2501, USA*

[5]*CNR, IENI, Via Roberto Cozzi 53, 20125 Milano, Italy*

(Received 17 April 2014; revised manuscript received 19 August 2014; published 17 September 2014)

The statistical properties of fracture strength of brittle and quasibrittle materials are often described in terms of the Weibull distribution. However, the weakest-link hypothesis, commonly used to justify it, is expected to fail when fracture occurs after significant damage accumulation. Here we show that this implies that the Weibull distribution is unstable in a renormalization-group sense for a large class of quasibrittle materials. Our theoretical arguments are supported by numerical simulations of disordered fuse networks. We also find that for brittle materials such as ceramics, the common assumption that the strength distribution can be derived from the distribution of preexisting microcracks by using Griffith's criteria is invalid. We attribute this discrepancy to crack bridging. Our findings raise questions about the applicability of Weibull statistics to most practical cases.

DOI: 10.1103/PhysRevApplied.2.034008

I. INTRODUCTION

The applicability of the Weibull distribution to describe the fracture strength of brittle and quasibrittle materials has been a topic of intense debate [1–9]. Several experimental studies argue that the Weibull distribution is not always the best statistical distribution to fit fracture data [1,2,6,8,10–12] (numerous others argue otherwise), particularly for quasibrittle materials that have significant precursor damage. These observations demand a general theoretical explanation. The suggested explanations for these empirical observations include bimodal or multimodal flaw-size distribution [1,13–15], R-curve behavior [6], small size of the data sets [1,9], and thermal-activated crack nucleation [16,17]. Here we provide a general explanation for these observations by showing that the Weibull distribution is unstable in the renormalization-group sense for quasibrittle materials and, thus, not applicable at long length scales.

In deriving the Weibull distribution of fracture strengths, it is invariably assumed that the material volume has a population of noninteracting cracklike defects, and fracture happens as soon as the weakest of these defects starts to grow [18–20]. This assumption is also known as the "weakest-link hypothesis." Experimental observations suggest that this assumption does not hold for a large class of quasibrittle materials. These materials, including paper [21], granite [22,23], antler bone [24], wood [23,25], and composites [26,27], etc., typically "crackle" [5,28], suggesting that several local cracks grow and get arrested prior to global fracture. Advanced composites are designed to fail gracefully; that is, they have multiple failures before the ultimate fracture. It is clear that for such materials the weakest defect does not dominate the fracture properties of the material, and the defects interact via elastic fields. The emergent scale-invariant properties of these interactions have been a topic of intense study in the statistical physics community [29–32]. Several researchers have used the Weibull theory to model these quasibrittle materials. We show that even if the microscopic strength distribution is Weibull, the emergent distribution is significantly distorted due to elastic interactions and metastability. Thus, the Weibull distribution is not stable in the renormalization-group sense. We provide numerical evidence to support our theoretical claims.

For brittle materials such as glasses or ceramics that fracture catastrophically without precursor damage, it is assumed that the distribution of fracture strength can be derived from the distribution of flaw sizes by using Griffith's criteria (or, equivalently, the stress-intensity approach) and ignoring effects such as crack bridging or coalescence [1,33]. For exponentially distributed cracks, the fracture strength is expected to be described by the Duxbury-Leath-Beale (DLB) distribution [5,34,35], while only in the case of power-law-distributed cracks, one expects to obtain the Weibull distribution [36]. It is, however, challenging to measure the flaw-size distribution experimentally and, thus, these assumptions are rarely verified empirically [14,37,38]. One of the aims of this paper is to use numerical simulations to show that the

*stefano.zapperi@cnr.it

simple relations that are widely used in the literature are not accurate, and further study is needed to understand the discrepancy. This observation has important implications for material engineers who aim to improve the fracture properties of brittle materials by controlling the microstructure.

In light of the above-discussed limitations of the Weibull theory, what distribution should one use to fit fracture data? To answer this question, we consider two classes of fuse networks to model brittle and quasibrittle materials. Both of the these models are derived from the classical fuse-network models [34,39]. In the model for brittle materials, the fuse network is seeded with power-law-distributed cracks with varying morphology. This is different from the classical diluted-fuse-network model which has an exponential distribution of cracks [5,34,35]. The model for quasibrittle materials has a continuous distribution of fuse strengths, where each fuse strength is a random number drawn from a standard Weibull distribution. In this manner, we can ensure that the microscopic strength distribution is Weibull and study the emergent macroscopic distribution. This model differs from its counterparts in the literature [29,40] by the choice of the microscopic disorder and enables a numerical study of the stability of the Weibull distribution. Analyzing the simulations, we find that the recently proposed T method (T denotes transformation) provides a suitable alternative to fit the numerical data [41]. The method is general enough that it can be applied in a wide variety of cases.

The rest of the paper is organized as follows. Section II presents the basics of the classical Weibull theory and the commonly used relation between the strength distribution and the defect-size distribution. The details of the numerical models used in this study are discussed in Sec. III. Section IV presents theoretical and numerical evidence to show that the Weibull distribution is unstable under coarse graining for quasibrittle materials. In Sec. V, we present the numerical evidence to show that the relation between the strength distribution and the flaw-size distribution is nontrivial and cannot be obtained by a straightforward application of Griffith's criteria. We discuss the possible sources of the observed discrepancy. Section VI presents a comparison of the performance of the Weibull distribution and the recently proposed T method for fitting the simulation data for quasibrittle fuse networks. The conclusions are presented in Sec. VII.

II. WEIBULL THEORY

In this section, we review the classical Weibull theory in order to facilitate the discussion in the following sections. We consider a material volume V subjected to a stress field $\sigma(r)$. The material should have a density of defects of various shapes and sizes, such that $e^{-f(\sigma)}$ is the probability of not finding a defect with critical stress less than σ in a volume V_0 of the material. Here we assume that the stress in uniaxial and tensile; the case of full tensorial stress is similar and is not presented here to avoid unnecessary notational complexity. The volume V_0 should be large enough that it contains a sufficient number of cracks and yet small enough that the stress can be considered roughly constant across it; it is sometimes also called the representative volume element. $f(\sigma)$ is a homogeneous material property. Then, the probability that the material volume V will survive the stress field $\sigma(r)$ is given by

$$S_V(\sigma) = \exp\left(-\frac{1}{V_0}\int_V f(\sigma(r))dr^3\right). \tag{1}$$

Weibull recognized that taking $f(\sigma) = (\sigma/\bar{\sigma})^k$, where $\bar{\sigma}$ is a material-dependent scale parameter and k is the material-dependent Weibull modulus, gave a good fit for several brittle materials and introduced what is now known as the standard Weibull distribution [18]:

$$S_V(\sigma) = \exp\left[-\frac{1}{V_0}\int_V \left(\frac{\sigma}{\bar{\sigma}}\right)^k dr^3\right]. \tag{2}$$

It turns out that the empirical choice made by Weibull can be justified by a renormalization-group calculation in which one writes recursive equations describing the failure distribution as the scale is changed [42]. The Weibull distribution is one of the possible fixed points of the renormalization-group transformation [42].

The Weibull distribution can alternatively be derived by connecting the function $f(\sigma)$ to the microscopic-defect-size distribution. The basic calculation outlined in the remainder of this section can be found in a number of important references [1,33]. According to Griffith's criteria, a crack of length w is stable at applied normal stress σ if

$$K = \sigma Y w^{1/2} \leq K_{Ic}, \tag{3}$$

where Y is the geometry factor of the crack, and K_{Ic}, the critical stress-intensity factor, is a material property. The exponent of $1/2$ is applicable for ideally sharp cracks and can have a different value for wedge-shaped or blunted cracks. Thus, if we take $e^{-h(w)}$ to be the probability that the volume element V_0 does not contain any crack longer than w, we have

$$f(\sigma) = h(K_{Ic}^2/\sigma^2 Y^2). \tag{4}$$

If the defect-size (crack-length) distribution is a power law with exponent γ, then $h(w) \sim w^{-\gamma}$, which gives $f(\sigma) \sim (\sigma/\bar{\sigma})^{2\gamma}$, where $\bar{\sigma} = K_{Ic}/Y$. Thus, a power-law defect-size distribution with exponent γ leads to a Weibull distribution of fracture strength with modulus $k = 2\gamma$.

As pointed out before, this entire analysis assumes that the flaws do not interact and that the failure of the weakest flaw leads to the failure of the entire material volume. We also show in Sec. IV that relaxing the assumption that the weakest flaw leads to global failure has important consequences and results in the strength distribution flowing away from the Weibull form. Our numerical calculation reported in Sec. V shows that crack bridging is an important form of crack interaction that can significantly alter the resulting Weibull modulus away from the dilute limit (but does not change the Weibull form for power-law-distributed cracks).

III. THE RANDOM-FUSE MODEL

In this section, we describe the computational model that we use for various classes of brittle and quasibrittle materials. The theoretical arguments presented in later sections benefit from having a concrete model as a point of reference. We study several variants of the basic two-dimensional random-fuse model (RFM) [34,39]. The RFM is a well-accepted model of brittle fracture where each fuse represents a coarse-grained material region (analog of the classical representative volume element). The model consists of a set of conducting fuses with unit conductivity $g_j = 1$ and breaking threshold σ_j arranged on a 45°-tilted square lattice composed by $L \times L$ nodes. A unit voltage drop is applied along two parallel edges of the lattice while periodic boundary conditions are imposed along the other two edges. The Kirchhoff equations are solved numerically using the algorithm proposed in Ref. [43] to determine the current flowing in each of the fuses. We then evaluate the ratio between the current i_j and the breaking threshold σ_j, and the fuse having the largest value $\max_j \frac{i_j}{\sigma_j}$ is irreversibly removed (burned). The current is redistributed instantaneously after a fuse is burned. Each time a fuse is burned, it is necessary to recalculate the current distribution in the lattice. The process of burning fuses, one at a time, is repeated until the lattice system fails completely (becomes nonconductive). The random-fuse model is equivalent to a scalar elastic problem where we consider a pure antiplane shear deformation. In this condition, the shear stress σ is related to the total current I by $\sigma = I/L$, the shear strain ϵ to the voltage drop v by $\epsilon = v/L$, and the conductivity g is equivalent to the shear modulus. From the breaking sequence, we can derive the current-voltage (or stress-strain) curve of the network under adiabatic loading as discussed in Ref. [44].

In this study, we employ two different disorder distributions to model quasibrittle and brittle materials:

(i) Weibull disorder (quasibrittle). The fuse strength threshold is chosen to be a random variable drawn from a Weibull distribution with modulus k; thus, the survival probability of a fuse at applied stress σ is $S_1(\sigma) = e^{-\sigma^k}$. Fuse networks with continuously distributed strengths have been studied previously [29]. In those studies, the thresholds were drawn from the uniform [29], power-law [30,31], and hyperbolic distributions [45]. However, the focus of those studies was on the morphology and dynamic properties, while we focus on strength. Further, by letting the local thresholds be Weibull distributed, we can directly study the stability of the Weibull distribution under coarse graining.

(ii) Diluted cracks (brittle). We remove a fraction p of the fuses and assign the same breaking threshold ($= 1$) to the intact fuses [5,34]. We take $0.05 \leq p \leq 0.2$, thus, keeping the initial damage fairly dilute in order to avoid the phenomena that happen near the percolation threshold (at $p = 0.5$ for the tilted square lattice we are using). Note that the missing fuses are not chosen randomly but rather in a way that they form a set of cracks with power-law-distributed crack lengths with $2.5 \leq \gamma \leq 9$, where γ is the exponent of the power law. We employ both straight and fractal flaws grown by using self-avoiding random walks. Fuse networks with diluted cracks were originally studied in Refs. [34,39]. However, in those studies the cracks' lengths had an exponential distribution (as opposed to power law). The exponential distribution of defect sizes leads to a Gumbel-type distribution of strengths and, thus, are markedly different from our model.

For each case, we do extensive statistical sampling for network sizes $L = 32$, 64, 128, 256, 512.

IV. STABILITY OF WEIBULL DISTRIBUTION FOR QUASIBRITTLE MATERIALS

The standard Weibull distribution reported in Eq. (2) is derived under the assumption that the failure of the weakest flaw (or representative volume element) leads to complete global failure. Under this assumption, if the strength distribution of the representative element is standard Weibull with modulus k, i.e., $S_{V_0}(\sigma) = e^{-\sigma^k}$, then the survival probability of the material volume V is given by

$$S_V(\sigma) = S_{V_0}(\sigma)^{V/V_0} = \exp\left(-\frac{V}{V_0}\sigma^k\right). \tag{5}$$

In mathematical terms, we can say that the Weibull distribution is stable under coarse graining: A system composed by subsystems described by the Weibull distribution is itself described by the Weibull distribution.

As we mentioned earlier, however, the weakest-link assumption is not accurate in quasibrittle materials. We can then derive the condition for Eq. (5) to remain valid if this assumption is relaxed. The stress at which the weakest flaw fails scales as $\sigma_{\min} \sim (V_0/V)^k$. The failure of this volume element enhances the stress on its neighbors due to stress concentration. However, the neighbors of the weakest flaw are typically not very weak, and we safely assume that their strength is near the mean strength $\langle\sigma\rangle = 1$. Assuming that the stress concentration factor scales as YV_0^β, where Y is a geometry factor, the neighboring volume element fails if $\sigma_{\min} YV_0^\beta > \langle\sigma\rangle$, which yields $k \gtrsim \log(V/V_0)$ as the approximate condition for the validity of the weakest-link hypothesis. Outside of this range, the local failure of the weakest link does not trigger global failure. In this calculation, we ignore the details and make several simplifications; thus, it gets only the correct scaling.

The above arguments show that the weakest-link hypothesis is self-consistent, and the Weibull distribution is stable under coarse graining only if the Weibull modulus is large enough, $k \gtrsim \log(V/V_0)$. Clearly, the strength distribution flows away from the Weibull distribution in the limit of $V \to \infty$. The typical ranges for the Weibull modulus are $k > 30$ for metals, $5 < k < 20$ for ceramics [46], and $2 < k < 4$ for biomaterials such as nacre [47]. It is clear that for materials with small to moderate values of k (such as biomaterials), the applicability of Weibull analysis is questionable. Indeed, the weakest-link hypothesis is manifestly false—these materials exhibit significant precursory fracture events (crackling noise) before failure [28].

Figure 1 shows the emergent strength distribution for fuse networks of various sizes where the fuse threshold is taken from a Weibull distribution with $k = 25$. We choose such a high value of k to show the crossover away from Weibull; for smaller values of k, the distribution has already flown away from Weibull even for the smallest networks that we can simulate. According to the Weibull theory, the emergent distribution of strength would be given by Eq. (5) with $V/V_0 = L^2$ (there are L^2 fuses), thus, giving $S_{L^2}(\sigma) = e^{-L^2\sigma^k}$. Figure 1 shows that while this prediction holds for small values of L, the distribution flows away from the Weibull distribution at longer lengths. This shows that the Weibull distribution is unstable to disorder in a renormalization-group sense and must be used with caution for quasibrittle materials.

We establish that the strength distribution flows away from Weibull in quasibrittle materials, but what does it flow towards? It is an unsolved problem to compute the new emergent distribution of strengths analytically. However, to get some idea about the distribution, we compute a very simple-minded upper bound to the survival probability for the fuse-network model. From Eq. (3), at any given stress σ, the length of the critical crack goes as $w_{\mathrm{cr}}(\sigma) \sim (\bar{\sigma}/\sigma)^2$ (i.e., a crack longer than w_{cr} will have unstable growth). If the fuse strength threshold is standard Weibull, then the probability of having a crack of size w_{cr} at any given lattice site is at least $(1 - e^{-\sigma^k})^{w_{\mathrm{cr}}(\sigma)}$. Since there are L^2 lattice sites, the global probability of survival is at most

$$[1 - (1 - e^{-\sigma^k})^{w_{\mathrm{cr}}(\sigma)}]^{L^2}. \tag{6}$$

Making asymptotic expansions for small σ, we get

$$S_{L^2}(\sigma) < \exp(-L^2 e^{-k(\bar{\sigma}/\sigma)^2 \log(1/\sigma)}). \tag{7}$$

If we take the slowly varying $\log(1/\sigma)$ to be a constant, then the above expression is reminiscent of DLB distribution [34]. The factor of $\log(1/\sigma)$ can be removed in a more natural way if one takes into account the stress concentration at each step of crack growth (see Ref. [40] for a

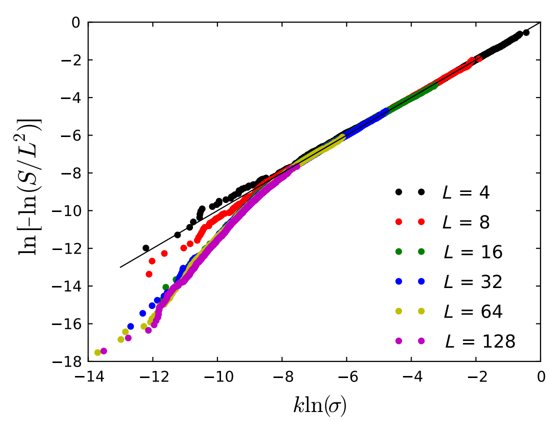

FIG. 1. Emergent survival probability for $L = 4, \ldots, 128$, with $k = 25$ for the threshold distribution. If the emergent distribution is Weibull, it will follow the solid black line. Clearly, the distribution flows away from Weibull at long length scales, showing that the Weibull distribution is not stable in a renormalization-group sense.

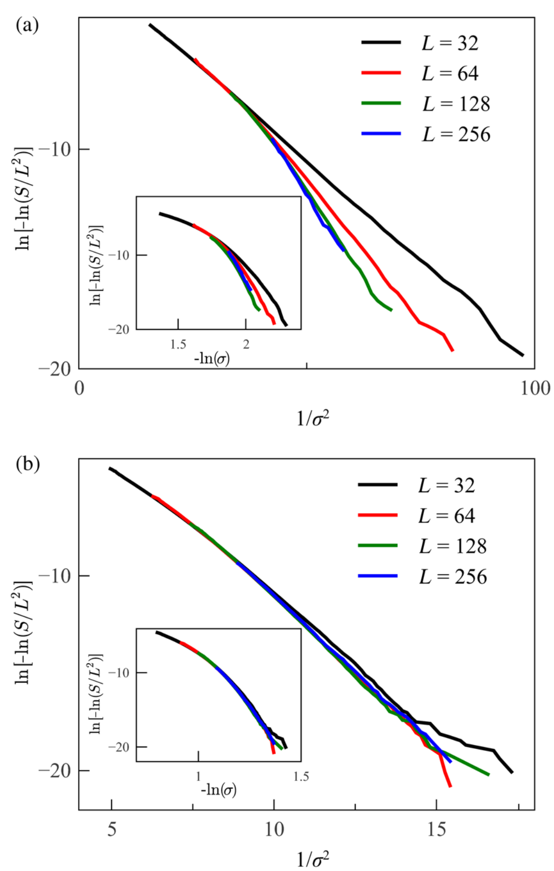

FIG. 2. Survival probability for (a) $k = 1.5$ and (b) $k = 4$. The main figures show the DLB test, while the insets show the Weibull test; straight lines indicate agreement with the tested form.

similar treatment). Our observation is supported by experimental results for some quasibrittle materials where the DLB distribution was found to fit the data better than the Weibull distribution [10–12].

Since the upper bound that we establish decays faster than any Weibull function at $\sigma = 0$, the macroscopic survival probability cannot be of the Weibull form, even if the microscopic distribution is Weibull. Note that the arguments made here are fairly general, and, thus, we expect the macroscopic strength distribution for any material with significant precursor damage to deviate from the Weibull distribution. We confirm that these ideas are consistent with the results of our numerical simulations. Figure 2 shows the survival probability obtained by statistical sampling of fuse networks with different values of k. The main plot in the figure shows that the survival probability is consistent with a DLB distribution. If instead the survival probability is consistent with a Weibull distribution, then the insets in the figure (so-called Weibull plots) will be straight lines. However, the plots show considerable curvature, suggesting a deviation from the Weibull distribution at long length scales.

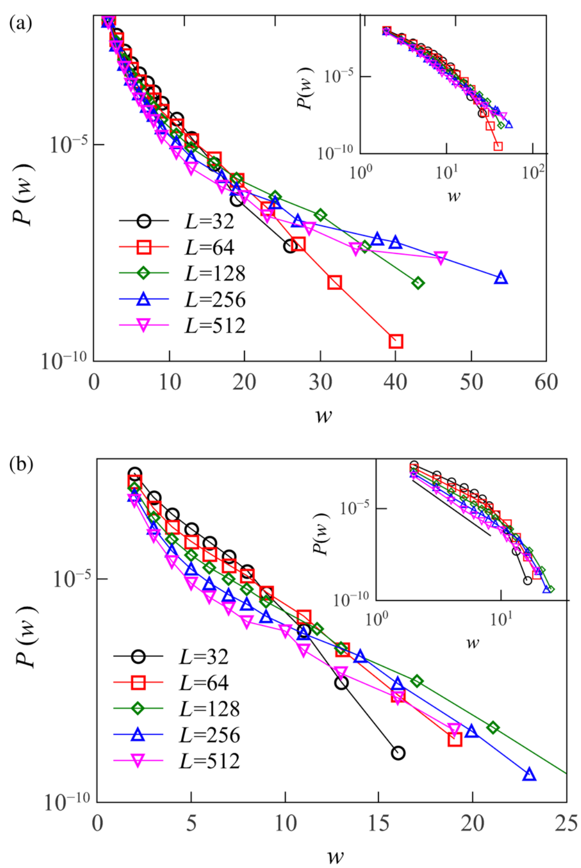

FIG. 3. Crack-width distributions at peak load for Weibull-distributed fuse strengths with exponents (a) $k = 1.5$ and (b) $k = 4$. The distribution is a power law with an exponential tail for all values of k.

The DLB distribution was originally associated with samples having an exponential distribution of crack length [5,34]. We confirm this hypothesis in our simulations by measuring the crack-length distribution just before catastrophic failure. The result reported in Fig. 3 shows, indeed, the presence of an exponential tail.

V. DEFECT DISTRIBUTION AND WEIBULL MODULUS FOR BRITTLE MATERIALS

It is widely assumed that the emergent Weibull modulus for brittle materials can be derived by using Griffith's criteria if the crack-length distribution is known. This assumption has been used in several important studies [1,33]. However, it has never been verified empirically due to experimental challenges. We examine this assumption numerically by simulating fuse networks seeded with power-law-distributed cracks. Cracks are created by removing a certain fraction p of fuses from the network. The net density of cracks p is kept low (< 0.2) to mimic materials such as glasses or ceramics where the density of microcracks is small. The critical effects associated with approaching the percolation threshold are also avoided by keeping p small. Unlike the classical fuse-network models, the removed fuses are chosen so as to generate a power-law distribution of crack lengths (Sec. III).

We derive the strength distribution based on the standard Griffith's-criteria-based assumption and compare the result to simulations. According to Griffith's theory, if the exponent of the power-law distribution of crack lengths is γ, then for Eq. (4) we have $h(w) \sim pw^{-\gamma}$ giving $f(\sigma) \sim p(\sigma/\bar{\sigma})^{2\gamma}$, where $\bar{\sigma} = K_{Ic}/Y$. This yields the following Weibull distribution of strengths for a fuse network of linear size L and "volume" L^2 (assuming uniform stress):

$$S_{L^2}(\sigma) = e^{-L^2 p(\sigma/\bar{\sigma})^{2\gamma}}. \tag{8}$$

Thus, the Weibull modulus is given by $k = 2\gamma$.

The above discussion assumes that flaw distribution does not change at all in the fracture process. In real materials, as well as in our fuse-network model, there is at least a small amount of damage before catastrophic fracture. This damage can change the tail of the crack-width distribution. Let γ_i, γ_f be the exponent of the crack-size distribution before loading and at peak load, respectively. We investigate the relation between γ_i, γ_f, p, and k numerically. We find in our simulations that $\gamma_i < \gamma_f$. Further, we find that the modulus of the emergent Weibull distribution is related to the damage distribution at peak load, $k = 2\gamma_f$. Figure 4(a) shows the comparison of the crack-size distribution at zero and peak load for $\gamma_i = 5$. Figure 4(b) shows the corresponding survival probability on a so-called Weibull plot. The slope of the Weibull plots agrees well with $2\gamma_f$.

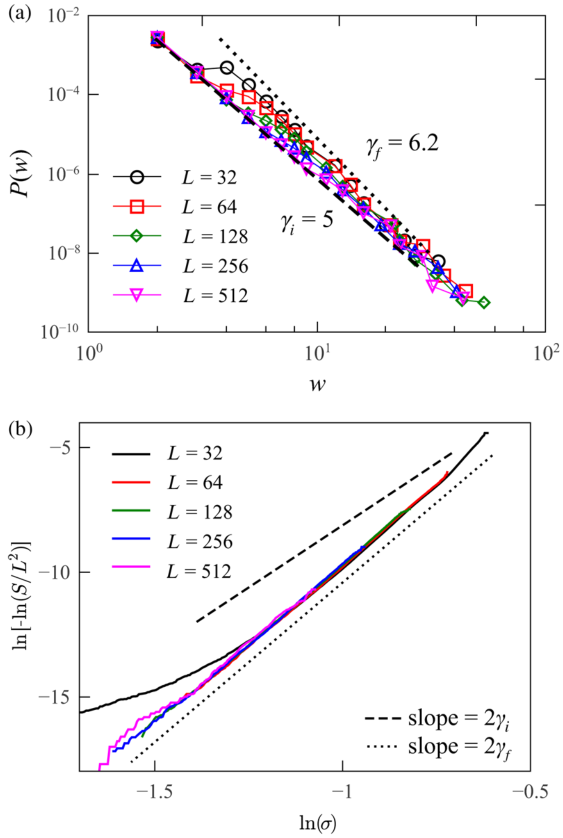

FIG. 4. (a) Crack-width distributions at peak load for a system with power-law-distributed cracks. The power-law tail has an exponent γ_f that is larger than the initial one γ_i. (b) The corresponding survival distribution obeys the Weibull law with $k = 2\gamma_f$.

Thus, the standard assumption of $k = 2\gamma_i$ is incorrect. We further explore the relation between γ_i and γ_f by carrying out extensive numerical simulations for $2.5 < \gamma_i < 9.0$ and $0.01 \le p \le 2$. We also investigate the effect of the shape of initial cracks. We seed the network either with straight cracks or fractal-looking cracks grown by using self-avoiding random walks. In both cases, we maintain the width distribution $h(w) \sim w^{-\gamma_i}$ and the defect density as dictated by p. Figure 5 shows the relation between γ_i and γ_f for various values of p for straight as well as grown cracks. For all the cases, we observe that $\gamma_f > \gamma_i$. It is reasonable to expect a slight increase in the exponent γ due to crack bridging. However, it is not clear what causes the almost 3-times increase in γ for some configurations.

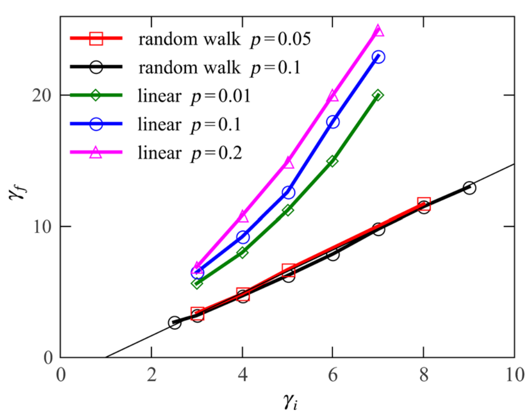

FIG. 5. Relation of the exponents of the crack-width distribution initially γ_{rmi} and at peak load γ_f. For linear-grown cracks, the relation depends strongly on p and γ_i, while for random-walk-grown cracks we find $\gamma_f \approx c\gamma_i + d$ for both investigated dilution parameters $p = 0.05$ and $p = 0.1$.

VI. *T* METHOD TO FIT THE STRENGTH DISTRIBUTION

Except for the case of power-law-distributed cracks, we see that the strength distribution is not Weibull and is probably of the DLB type. In a previous paper [5], we discussed how the extreme value functions are an extremely poor approximation to the DLB form. These considerations raise the following question: What form should be used to fit fracture data in practice?

One of the major concerns while fitting data to extreme-value distributions is the accuracy of extrapolations in the low-probability tail. We compare the standard Weibull theory and the recently proposed T method [41] by fitting the data fracture data for the quasibrittle fuse networks with the two techniques. In Weibull theory [Eq. (2)], the survival probability of the network is given by $S_{L^2}(\sigma) = e^{-L^2(\sigma/\bar{\sigma})^k}$. Given the observed data vector $\mathbf{X}$ (= vector of fracture strengths observed in simulation) of length n, the parameters $(\bar{\sigma}, k)$ are determined by using the maximum likelihood estimation as the values that maximize the following log-likelihood function:

$$\mathcal{L}_W(\bar{\sigma}, k|\mathbf{X}) = \sum_{i=1}^{n} \partial_{\mathbf{X}_i} \log[S_{L^2}(\mathbf{X}_i)]. \tag{9}$$

The parameters that minimize the above log-likelihood function give the best-fit parameters $(\bar{\sigma}, k)$ for the Weibull theory. The T method first applies a nonlinear transformation to the data, $T(\mathbf{X}) = \mathbf{X}^{-\alpha}$ and then fits the transformed data of an extreme value form, thus, giving the following log-likelihood function [41]:

$$\mathcal{L}_T(\alpha, a, b|\mathbf{X}) = \sum_{i=1}^{n} \log\{\partial_{\mathbf{X}_i} G_0[(T(\mathbf{X}_i) - b]/a\}, \tag{10}$$

where the parameters (α, a, b) are estimated by minimization, and $G_0(x) = \exp(-e^{-x})$ is the standard Gumbel distribution. We use the data set of over 20 000 simulations corresponding to the random-fuse model with $k = 1.5$ and $L = 128$ to test the applicability of the above method for such extrapolations. We choose 20 random samples of 200 data points from the data set. We then fit each of the smaller data sets using the Weibull theory and the T method.

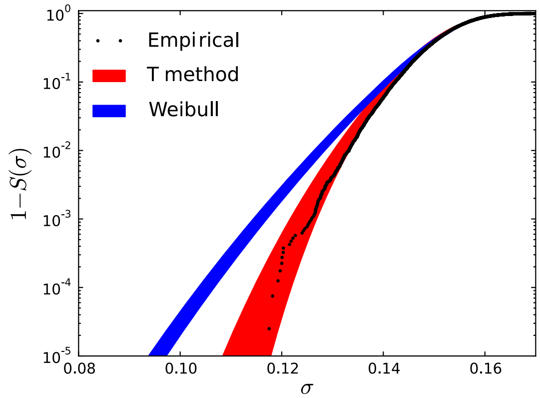

FIG. 6. Results of extrapolating the fits of the Weibull distribution and the suggested transformation-based method. Fits obtained from small data sets (size 200) by using the T method can be extrapolated with confidence to probabilities about 2 orders of magnitude smaller. The figure shows ± 1 standard deviation results.

We extrapolate the fits and compare predictions in the low-probability tail with the empirical data. Figure 6 shows the ± 1 standard deviation predictions of such fits. It is clear from the figure that the T method outperforms the standard Weibull theory in accuracy of the fit and extrapolation in the low-stress tail.

VII. CONCLUSIONS

In conclusion, we study the conditions for emergence of the Weibull distribution for fracture strength in brittle and quasibrittle materials. We show the Weibull distribution is unstable under coarse graining for a large class of materials where the weakest-link hypothesis is not strictly valid, and there is significant precursor damage. For the case of brittle materials, we show that the relation between strength distribution and the defect-size distribution is highly nontrivial and cannot be obtained by simple application of Griffith's criteria. Crack bridging has a significant effect on the tails of the crack-size distribution and, thus, changes the Weibull modulus considerably. We find that the recently proposed T method does a significantly better job at fitting the fracture strength data, as compared to the Weibull distribution. We hope that the our results will lead to further research and discussion about the applicability of the Weibull distribution for fracture data, particularly for quasibrittle materials that crackle.

ACKNOWLEDGMENTS

We thank Zoe Budrikis and Claudio Manzato for valuable comments and fruitful discussions. Z. B. and S. Z. are supported by the European Research Council through the Advanced Grant 2011 SIZEFFECTS. A. S. is partially supported by the Miller Institute for Basic Research in Science, Berkeley. J. P. S. and A. S. (partially) are supported by Grant No. DOE-BES DE-FG02-07ER46393. J. P. S. and S. Z. acknowledge support from Materials World Network: Cooperative Activity in Materials Research between U.S. Investigators and their Counterparts Abroad in Italy (NSF Grant No. DMR 1312160).

[1] R. Danzer, P. Supancic, J. Pascual, and T. Lube, Fracture statistics of ceramics—Weibull statistics and deviations from Weibull statistics, Eng. Fract. Mech. **74**, 2919 (2007).
[2] C. Lu, R. Danzer, and F. D. Fischer, Fracture statistics of brittle materials: Weibull or normal distribution, Phys. Rev. E **65**, 067102 (2002).
[3] R. H. Doremus, Fracture statistics: A comparison of the normal, Weibull, and type i extreme value distributions, J. Appl. Phys. **54**, 193 (1983).
[4] K. Trustrum and A. DeS. Jayatilaka, Applicability of Weibull analysis for brittle materials, J. Mater. Sci. **18**, 2765 (1983).
[5] C. Manzato, A. Shekhawat, P. K. V. V. Nukala, M. J. Alava, J. P. Sethna, and S. Zapperi, Fracture strength of disordered media: Universality, interactions, and tail asymptotics, Phys. Rev. Lett. **108**, 065504 (2012).
[6] Serkan Nohut and Chunsheng Lu, Fracture statistics of dental ceramics: Discrimination of strength distributions, Ceram. Int. **38**, 4979 (2012).
[7] Z. Bazant, Y. Xi, and S. Reid, Statistical size effect in quasibrittle structures: I. is Weibull theory applicable? J. Eng. Mech. **117**, 2609 (1991).
[8] Yevgeny Rozenblat, Dmitri Portnikov, Avi Levy, Haim Kalman, Sergej Aman, and Jürgen Tomas, Strength distribution of particles under compression, Powder Technol. **208**, 215 (2011).
[9] Bikramjit Basu, Devesh Tiwari, Debasis Kundu, and Rajesh Prasad, Is Weibull distribution the most appropriate statistical strength distribution for brittle materials? Ceram. Int. **35**, 237 (2009).
[10] I. C. van den Born, A. Santen, H. D. Hoekstra, and J. Th. M. De Hosson, Mechanical strength of highly porous ceramics, Phys. Rev. B **43**, 3794 (1991).
[11] M. Korteoja, L. I. Salminen, K. J. Niskanen, and M. J. Alava, Strength distribution in paper, Mater. Sci. Eng. A **248**, 173 (1998).
[12] C. Baxevanakis, D. Jeulin, and D. Valentin, Fracture statistics of single-fibre composite specimens, Compos. Sci. Technol. **48**, 47 (1993).
[13] K. Jakus, J. E. Ritter, T. Service, and D. Sonderman, Evaluation of bimodal concurrent flaw distributions, J. Am. Ceram. Soc. **64**, 372 (1981).
[14] N. Orlovskaja, H. Peterlik, W. Steinkellner, and K. Kromp, Prediction of strength of recrystallized siliconcarbide from pore size measurement part i the bimodality of the distribution, J. Mater. Sci. **35**, 699 (2000).
[15] H. Peterlik and D. Loidl, Bimodal strength distributions and flaw populations of ceramics and fibres, Eng. Fract. Mech. **68**, 253 (2001).

[16] S. Santucci, L. Vanel, and S. Ciliberto, Slow crack growth: Models and experiments, Eur. Phys. J. Spec. Top. **146**, 341 (2007).
[17] Loïc Vanel, Sergio Ciliberto, Pierre-Philippe Cortet, and Stéphane Santucci, Time-dependent rupture and slow crack growth: Elastic and viscoplastic dynamics, J. Phys. D **42**, 214007 (2009).
[18] W. Weibull, *A Statistical Theory of the Strength of Materials* (Generalstabens litografiska anstalts forlag, Stockholm, 1939).
[19] S. B. Batdorf and H. L. Heinisch, Weakest link theory reformulated for arbitrary fracture criterion, J. Am. Ceram. Soc. **61**, 355 (1978).
[20] M. Sutcu, Weibull statistics applied to fiber failure in ceramic composites and work of fracture, Acta Metall. **37**, 651 (1989).
[21] L. I. Salminen, A. I. Tolvanen, and M. J. Alava, Acoustic emission from paper fracture, Phys. Rev. Lett. **89**, 185503 (2002).
[22] D. A. Lockner, J. D. Byerlee, V. Kuksenko, A. Ponomarev, and A. Sidorin, Quasi-static fault growth and shear fracture energy in granite, Nature (London) **350**, 39 (1991).
[23] A. Garcimartin, A. Guarino, L. Bellon, and S. Ciliberto, Statistical properties of fracture precursors, Phys. Rev. Lett. **79**, 3202 (1997).
[24] P. Zioupos, J. D. Currey, and A. J. Sedman, An examination of the micromechanics of failure of bone and antler by acoustic emission tests and laser scanning confocal microscopy, Medical Engineering and Physics **16**, 203 (1994).
[25] A. Reiterer, S. E. Stanzl-Tschegg, and E. K. Tschegg, Mode I fracture and acoustic emission of softwood and hardwood, Wood science and technology **34**, 417 (2000).
[26] V. I. Yukalov, A. Moura, and H. Nechad, Self-similar law of energy release before materials fracture, J. Mech. Phys. Solids **52**, 453 (2004).
[27] J.-C. Anifrani, C. Le Floc'h, D. Sornette, and B. Souillard, Universal log-periodic correction to renormalization group scaling for rupture stress prediction from acoustic emissions, J. Phys. I (France) **5**, 631 (1995).
[28] James P. Sethna, Karin A. Dahmen, and Christopher R. Myers, Crackling noise, Nature (London) **410**, 242 (2001).
[29] S. Zapperi, P. Ray, H. E. Stanley, and A. Vespignani, First-order transition in the breakdown of disordered media, Phys. Rev. Lett. **78**, 1408 (1997).
[30] Alex Hansen and Jean Schmittbuhl, Origin of the universal roughness exponent of brittle fracture surfaces:stress-weighted percolation in the damage zone, Phys. Rev. Lett. **90**, 045504 (2003).
[31] Ashivni Shekhawat, Stefano Zapperi, and James P. Sethna, From damage percolation to crack nucleation through finite size criticality, Phys. Rev. Lett. **110**, 185505 (2013).
[32] Mikko J. Alava, Phani K. V. V. Nukala, and Stefano Zapperi, Statistical models of fracture, Adv. Phys. **55**, 349 (2006).
[33] A. DeS. Jayatilaka and K. Trustrum, Statistical approach to brittle fracture, J. Mater. Sci. **12**, 1426 (1977).
[34] P. M. Duxbury, P. L. Leath, and P. D. Beale, Breakdown properties of quenched random systems: The random-fuse network, Phys. Rev. B **36**, 367 (1987).
[35] P. D. Beale and P. M. Duxbury, Theory of dielectric breakdown in metal-loaded dielectrics, Phys. Rev. B **37**, 2785 (1988).
[36] A. M. Freundenthal, in *Fracture*, edited by H. Liebowitz (Academic Press, New York, 1968).
[37] Luen-Yuan Chao and Dinesh K. Shetty, Extreme-value statistics analysis of fracture strengths of a sintered silicon nitride failing from pores, J. Am. Ceram. Soc. **75**, 2116 (1992).
[38] Y. Zhang, Nozomu Uchida, Kazuyoshi Uematsu, Tadashi Hotta, K. Nakahira, and Masami Naito, A direct examination of the fracture strength-defect size relationship in alumina ceramics, Key Eng. Mater. **159–160**, 269 (1999).
[39] L. de Arcangelis, S. Redner, and H. J. Herrmann, A random fuse model for breaking processes, J. Phys. Lett. **46**, L585 (1985).
[40] B. Kahng, G. G. Batrouni, S. Redner, L. de Arcangelis, and H. J. Herrmann, Electrical breakdown in a fuse network with random, continuously distributed breaking strengths, Phys. Rev. B **37**, 7625 (1988).
[41] A. Shekhawat, Improving extreme value statistics, arXiv:1406.3372.
[42] G. Györgyi, N. R. Moloney, K. Ozogány, Z. Rácz, and M. Droz, Renormalization-group theory for finite-size scaling in extreme statistics, Phys. Rev. E **81**, 041135 (2010).
[43] P. K. V. V. Nukala and S. Simunovic, An efficient algorithm for simulating fracture using large fuse networks, J. Phys. A **36**, 11403 (2003).
[44] M. J. Alalva, P. K. V. V. Nukala, and S. Zapperi, Statistical models for fracture, Adv. Phys. **55**, 349 (2006).
[45] A. A. Moreira, C. L. N. Oliveira, A. Hansen, N. A. M. Araújo, H. J. Herrmann, and J. S. Andrade, Fracturing highly disordered materials, Phys. Rev. Lett. **109**, 255701 (2012).
[46] J. Tinschert, D. Zwez, R. Marx, and K. J. Anusavice, Structural reliability of alumina-, feldspar-, leucite-, mica- and zirconia-based ceramics, J. Dent. **28**, 529 (2000).
[47] R. Menig, M. H. Meyers, M. A. Meyers, and K. S. Vecchio, Quasi-static and dynamic mechanical response of haliotis rufescens (abalone) shells, Acta Mater. **48**, 2383 (2000).