compatible with F, then the three maps of the the canonical reduction $\rho = (f, g, h) : (C_n, d_n) \Rightarrow (C^c_n, d'_n)$ described in Theorem 7.1 are compatible with the filtrations.

Corollary 6.5 tells us that discrete vector fields can be used to speed up the computation of all persistent homology groups.

Corollary 7.4. *In the situation of Theorem 7.3, the map f of the reduction $\rho = (f, g, h) : (C_n, d_n) \Rightarrow (C^c_n, d'_n)$ induces isomorphisms $H^{p,b}_n(C_*) \cong H^{p,b}_n(C^c_*)$ for all $p \leq b$ in I and $M^{p,b}_n(C_*) \cong M^{p,b}_n(C^c_*)$ for all $p < b$ in I.*

Making use of this result and of Kenzo algorithms for computing admissible discrete vector fields [26] we have enhanced our programs computing multipersistence.

8 CONCLUSIONS AND FURTHER WORK

We presented a set of programs for performing computations on chain complexes with filtrations defined over posets. The programs allow to compute generalized persistent homology, and in particular some relevant invariants in the context of multipersistence. Although, due to the necessary adjustments to deal with infinite spaces, our programs are not as efficient as previous existing implementations with polynomial complexity, we provide algorithms which are valid in general situations, some of which cannot be tackled by any other method. One fundamental aspect of our implementation consists in the use of the effective homology technique, which makes it possible to handle infinitely generated chain complexes. Another important feature concerns the possibility of defining and using for computation filtrations over general posets. Our programs, improved using discrete vector fields, have been implemented as a new module for the Kenzo system.

We focused our study on filtrations indexed over the posets $\mathbb{Z}^m$ and $D(\mathbb{Z}^m)$, for their relevance in relation with multipersistence. In this respect, a theoretical contribution of our work is the description of the relation between persistent homology and spectral systems in a general scenario, which extends a result valid for persistent homology and spectral sequences arising from $\mathbb{Z}$-filtrations. Furthermore, we define and compute a new "empirical" invariant and show its discriminative power in the context of multipersistence.

Two fundamental requirements in persistent homology theory are computability and robustness. As a future research direction, we intend to reduce the computational cost for our invariants and to further investigate their behavior with respect to small changes in the multiparameter filtration. As we reviewed in Section 4, several approaches have been proposed to tackle the problems arising with multiparameter filtrations. Since effective homology displays a good behavior with respect to the invariants we considered in this work, studying its applicability to other constructions represents an interesting scope for further research.

ACKNOWLEDGMENTS

Partially supported by Basque Government, BERC 2018-2021 program; Spanish Ministry of Science, Innovation and Universities, BCAM Severo Ochoa accreditation SEV-2017-0718; Spanish Ministry of Science, Innovation and Universities, project MTM2017-88804-P; University of La Rioja - V Plan Riojano de I+D+I; Italian MIUR Award "Dipartimento di Eccellenza 2018-2022"- CUP: E11G18000350001 and SmartData@PoliTO center for Big Data and Machine Learning technologies.

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