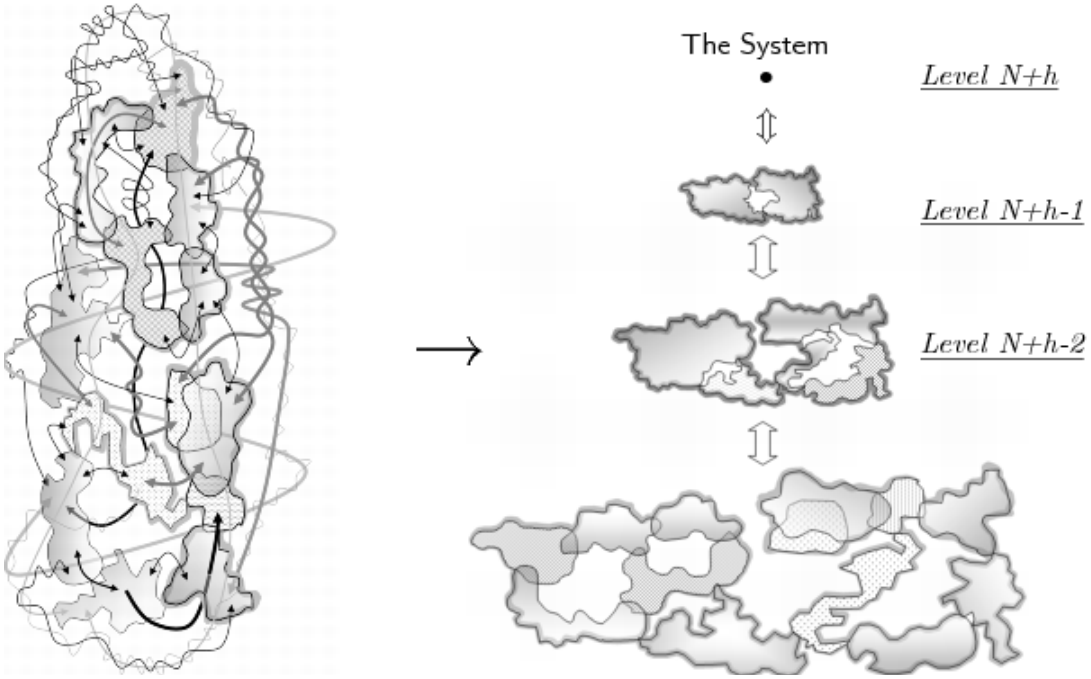
Beyond simplicial complexes: hypernetworks for systems of systems of systems



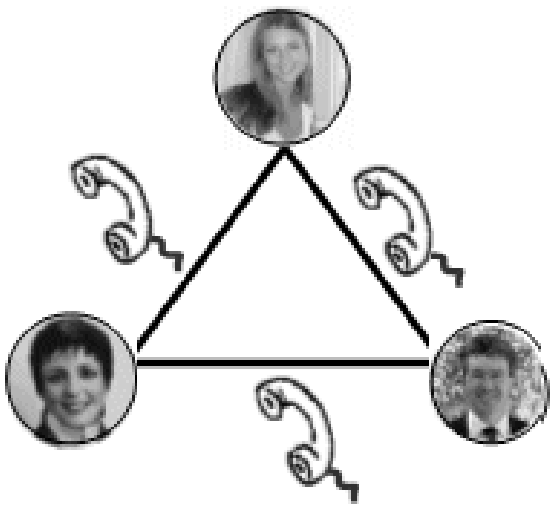
Jeffrey Johnson, The Open University, UK & ASSYST

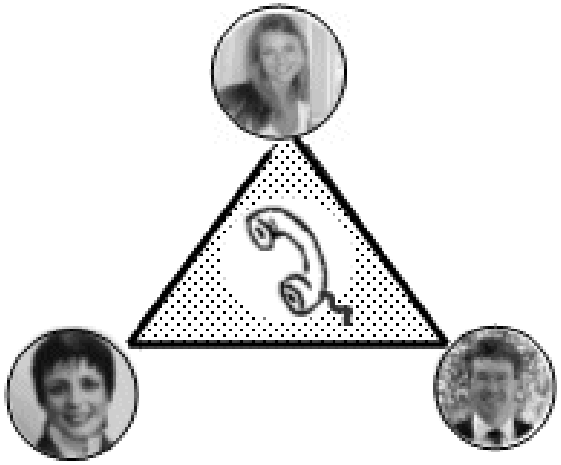
Part I : From networks to simplicial complexes

Part II: From simplicial complexes to hypernetworks

Part III: Issues in hypernetworks

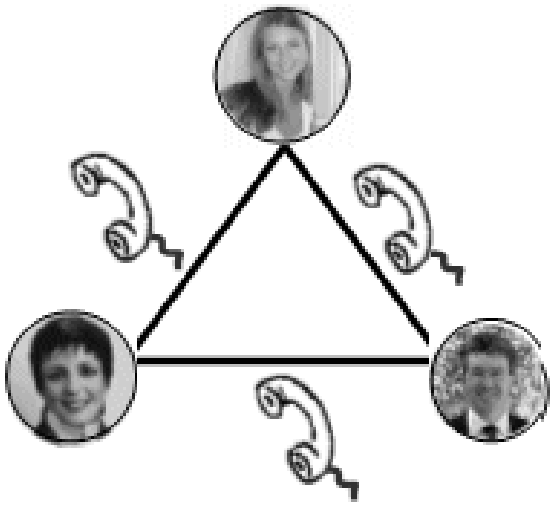
Binary relations are not rich enough

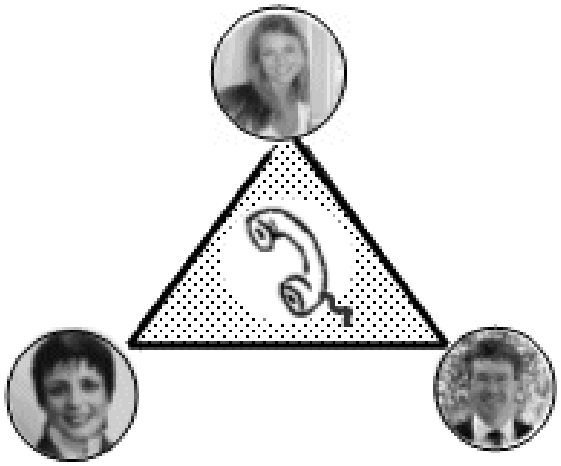


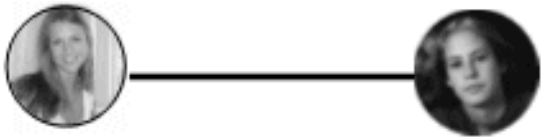


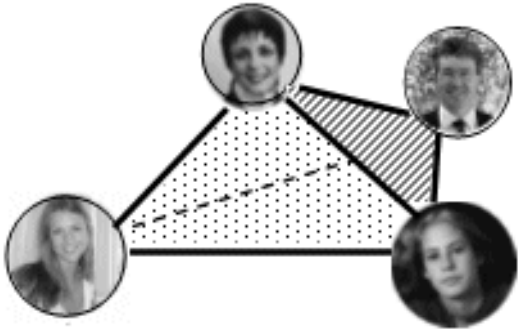
3 binary relations ≠ one 3-ary relation

Relational Structure









Binary relation **3-ary relation** **4-ary relation**

Relational Structure

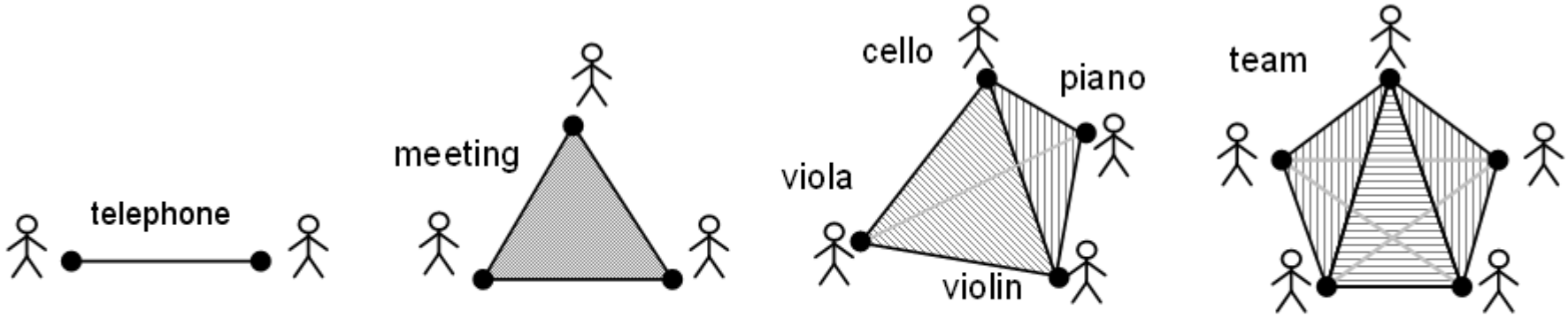
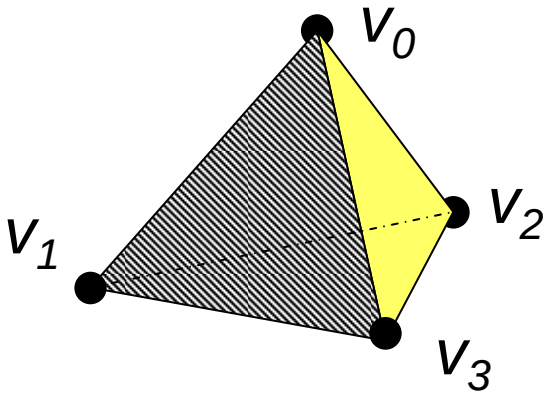


Figure 12. Representing relationships by multidimensional polyhedra

From networks to simplicial complexes

An abstract ***p-simplex*** is an ordered set of vertices,

$\sigma_p = \langle v_0, v_1, v_2, \ldots, v_p \rangle$.



$\sigma_3 = \langle v_0, v_1, v_2, v_3 \rangle$.

From networks to simplicial complexes

An abstract ***p-simplex*** is an ordered set of vertices,

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e.g. the tetrahedron

A *face* is a sub-simplex.

e.g. a triangle

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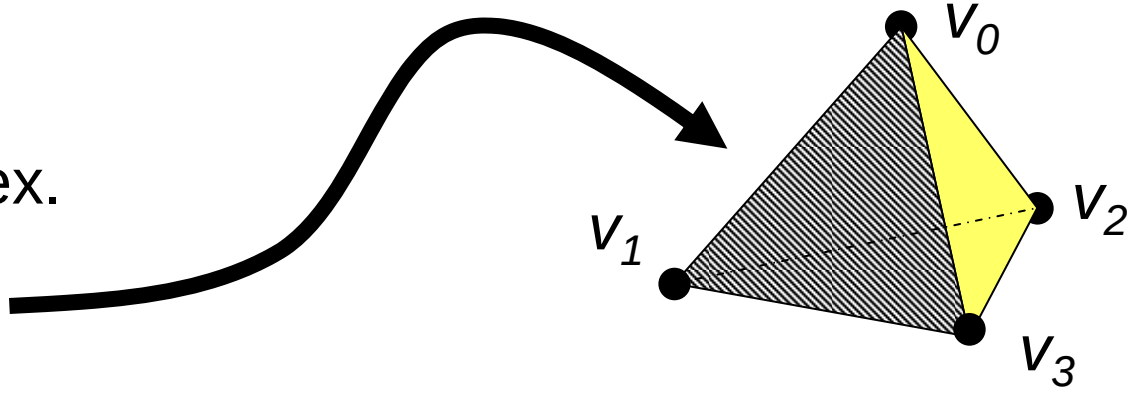

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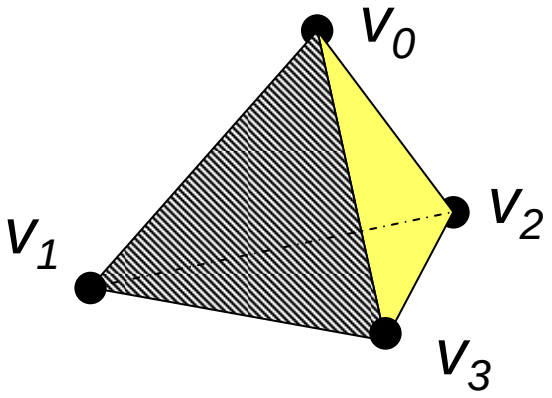


A *face* is a sub-simplex.

e.g. a triangle

$\sigma_3 = \langle v_0, v_1, v_2, v_3 \rangle$.

A *simplicial complex* is a set of simplices with all their faces

From networks to simplicial complexes

Every network is a simplicial complex whose simplices have dimension $q = 0$ or $q = 1$.

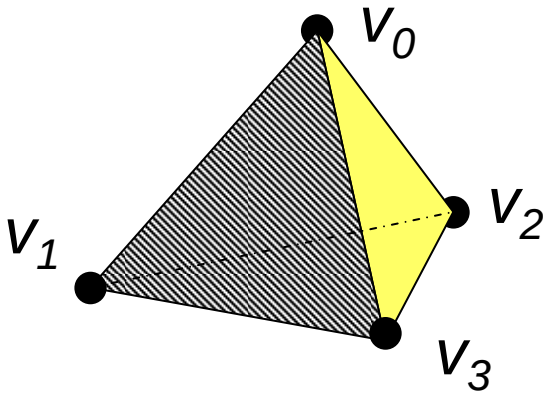
☞ **Simplicial complexes are a multidimensional generalisation of networks.**

From networks to simplicial complexes

An abstract ***p-simplex*** is an ordered set of vertices,

$\sigma_p = \langle v_0, v_1, v_2, \ldots, v_p \rangle$.

e.g. the tetrahedron



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e.g. a triangle

$\sigma_3 = \langle v_0, v_1, v_2, v_3 \rangle$.

A *simplicial complex* is a set of simplices with all their faces

From networks to simplicial complexes

Let K be a simplicial complex

A *chain* is an expression of the form

$$c = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 + n_4\sigma_4 + n_5\sigma_5 \qquad n_i \in \mathrm{J},\ \sigma_5 \in \mathrm{K}$$

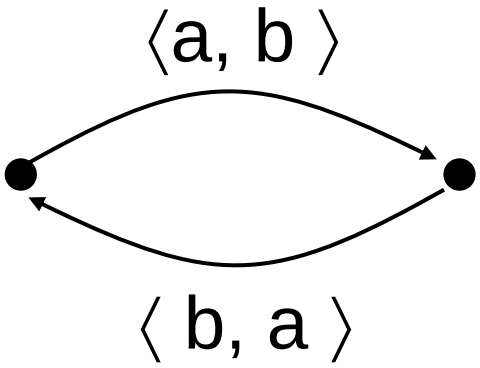
The set of all chains forms a module over the integers

K_p is the set of p-chains c_p, $dim(\sigma_i) = p \ \ \forall \ \sigma_i \ \text{in}\ c_p$

… more of this later

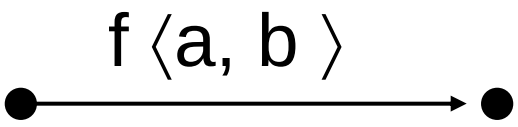
Simplicial complexes

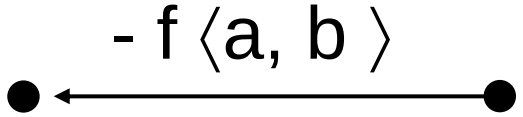
Networks are *directed*, $\langle a, b \rangle \neq \langle b, a \rangle$



Definition $\langle a, b \rangle = - \langle b, a \rangle$

Definition $f \langle a, b \rangle = - f \langle b, a \rangle$





Simplicial complexes

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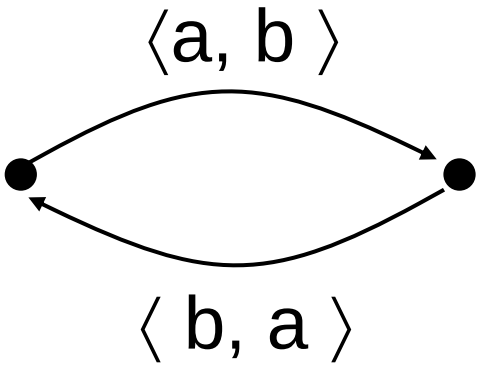
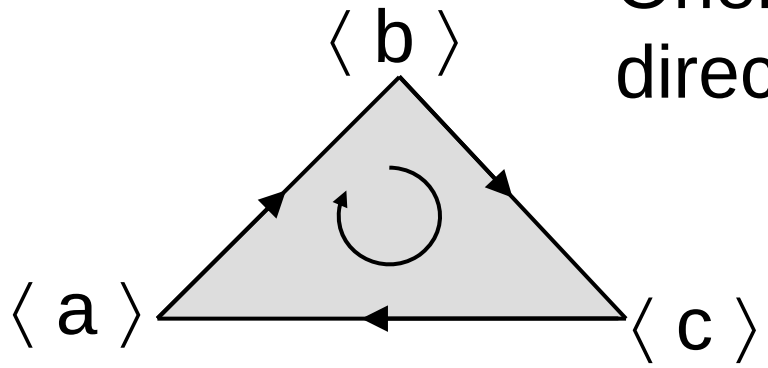
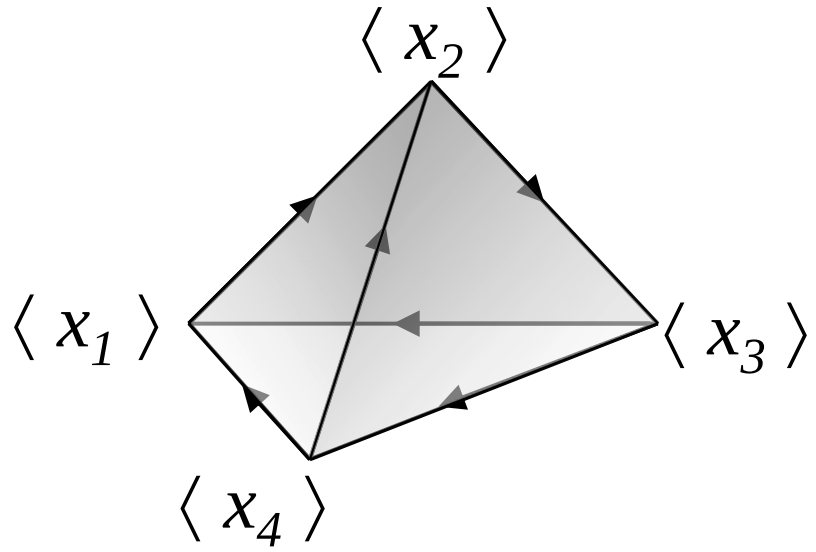




Orientation generalises directedness

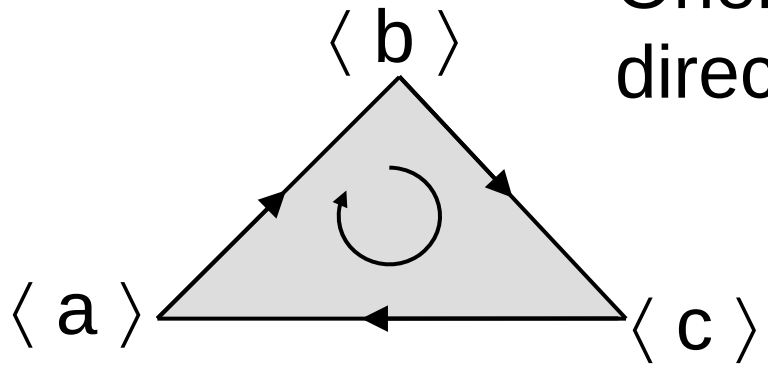
e.g. $\langle a, b, c \rangle = -\langle a, c, b \rangle$

Simplicial complexes



$$\langle \ldots, x_i, \ldots, x_j, \ldots \rangle$$

$$\overset{\text{def}}{=} -\langle \ldots, x_i, \ldots, x_j, \ldots \rangle$$



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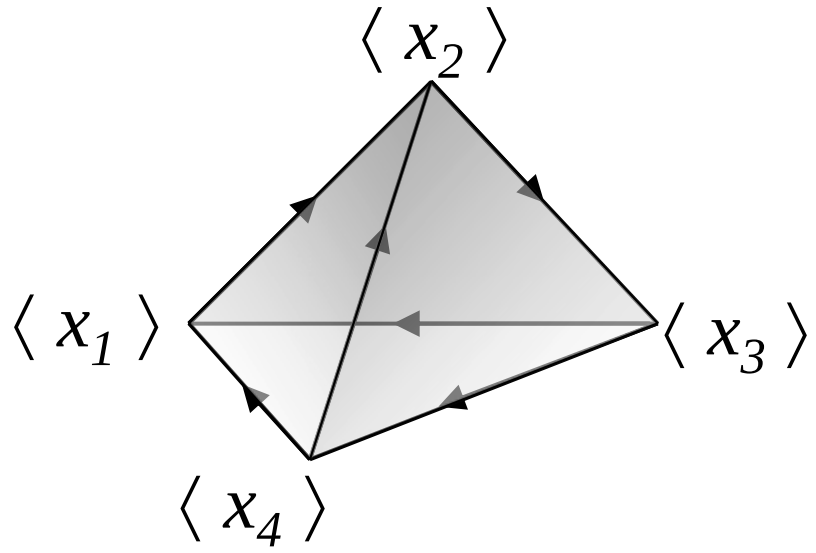
Simplicial complexes Why make these definitions?



$$\langle \ldots, x_i, \ldots, x_j, \ldots \rangle$$

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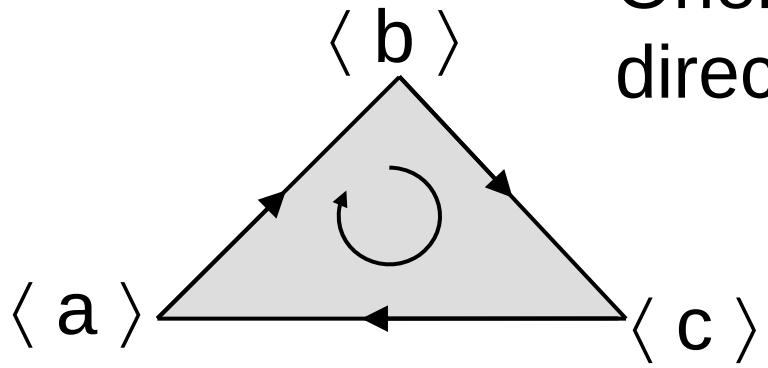


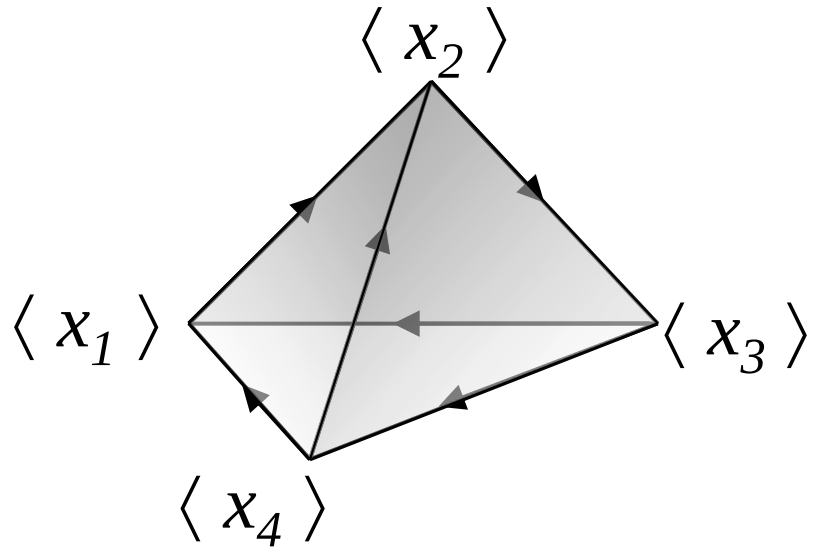
Orientation generalises directedness

e.g. ⟨ a, b, c ⟩ $\overset{\text{def}}{=}$ - ⟨ a, c, b ⟩

Simplicial complexes

Why make these definitions?

… to define *boundaries*



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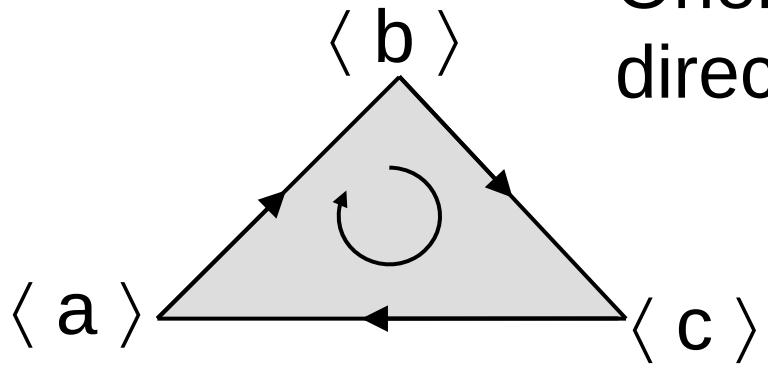

Orientation generalises directedness

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Simplicial complexes

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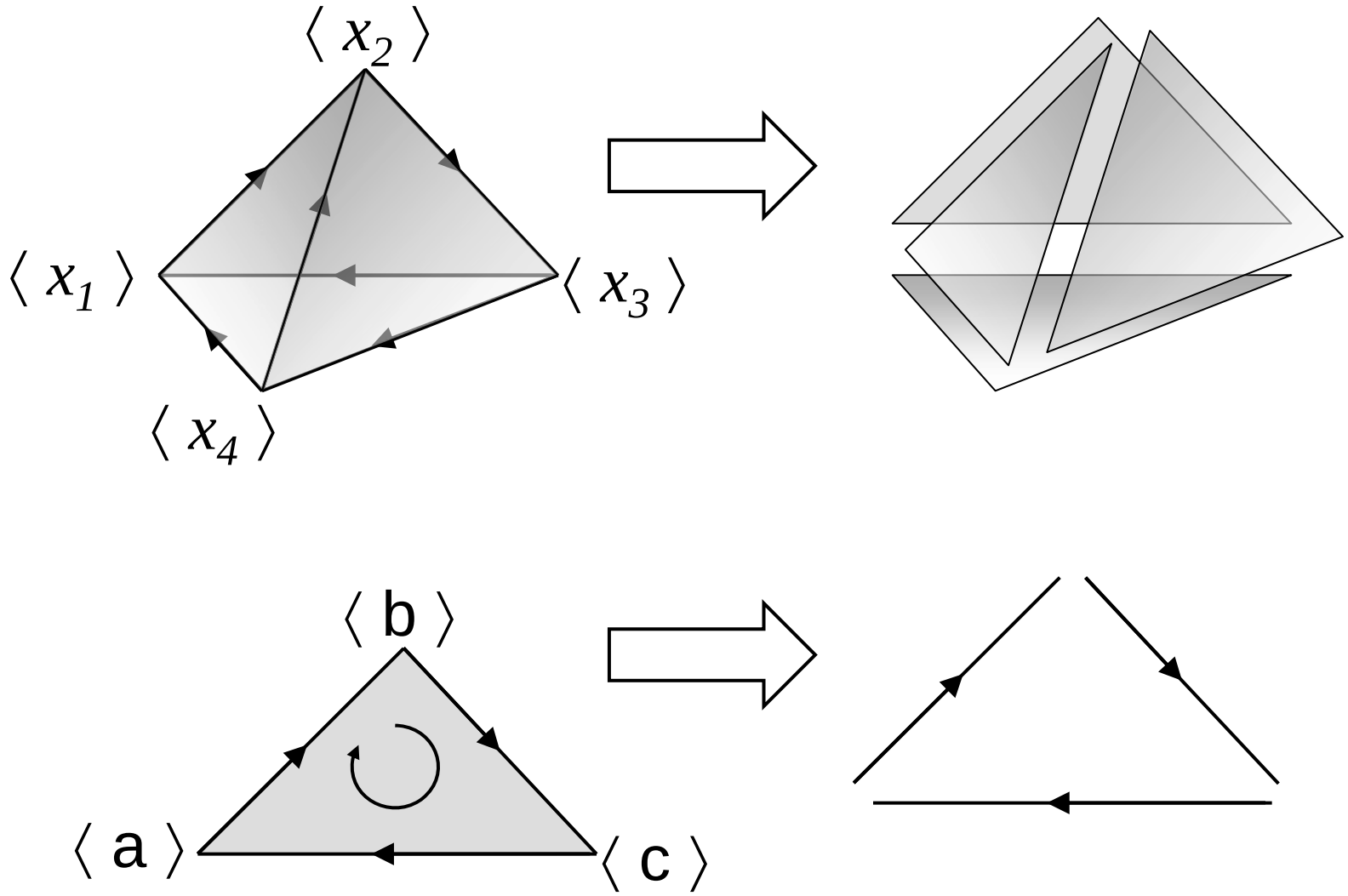


Simplicial complexes Why make these definitions?

… to define *boundaries*

The *boundary operator* ∂ is defined as

$$\partial\langle v_0, v_1, \ldots v_p\rangle = \Sigma_{i=0} \quad (-1)^i \langle v_0, \ldots, \widehat{v_i}, \ldots, v_p \rangle$$

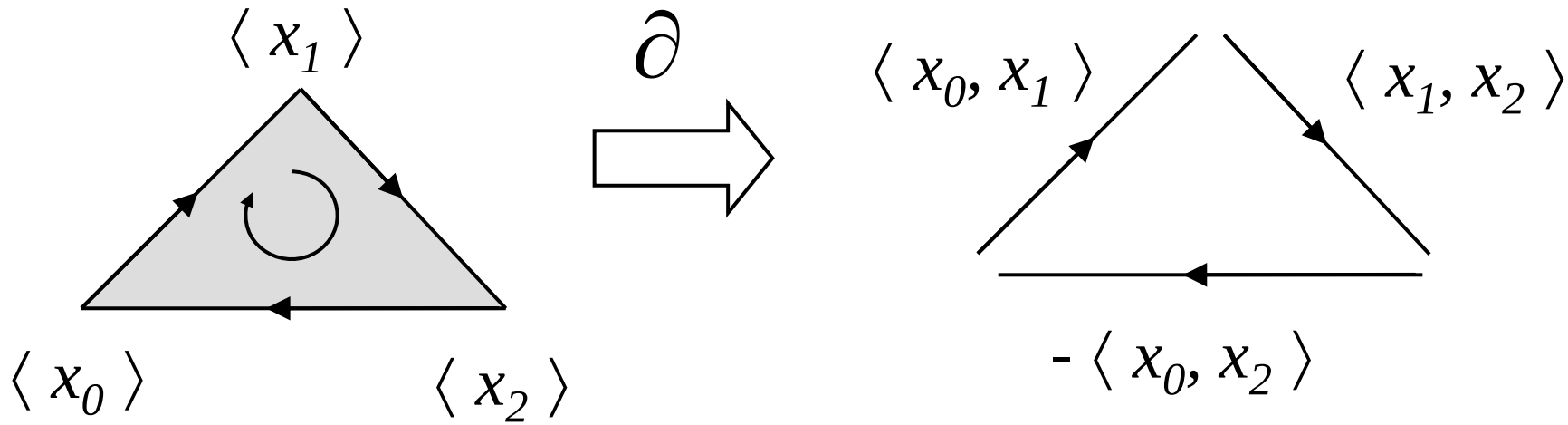
Where $\widehat{v_i}$ means v_i is removed.

Note – the boundary depends on orientation

… more of this later

Simplicial complexes Why make these definitions?

… to define *boundaries*



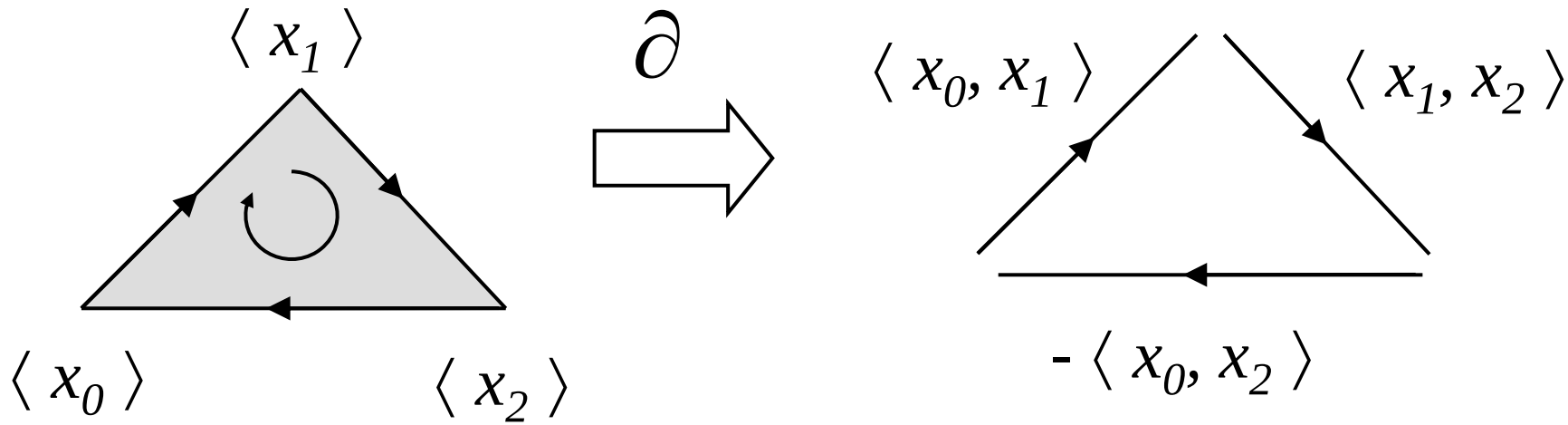
$$\partial\langle x_0, x_1, x_2 \rangle = \langle x_1, x_2 \rangle - \langle x_0, x_2 \rangle + \langle x_0, x_1 \rangle$$

Is the boundary of the triangle

Simplicial complexes Why make these definitions?

… to define *boundaries*

∂ is nilpotent, $\partial^2 = 0$



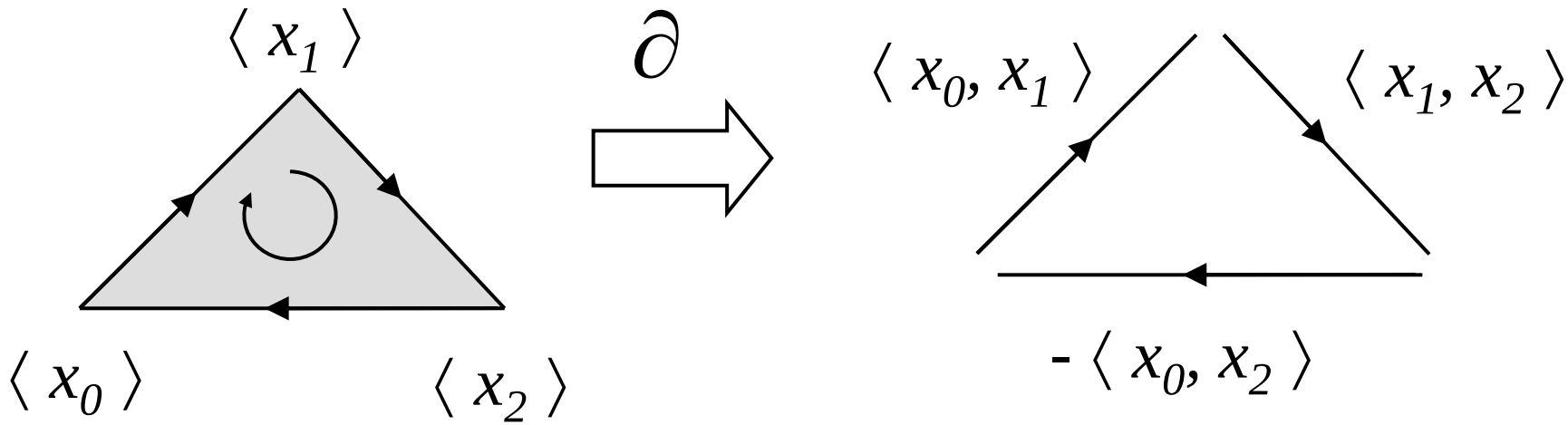
$$\partial^2\langle x_0, x_1, x_2 \rangle = \langle x_1, x_2 \rangle - \langle x_0, x_2 \rangle + \langle x_0, x_1 \rangle =$$

$$\langle x_2 \rangle - \langle x_1 \rangle - (\langle x_2 \rangle - \langle x_0 \rangle) + \langle x_1 \rangle - \langle x_0 \rangle$$

Simplicial complexes Why make these definitions?

… to define *boundaries*

∂ is nilpotent, $\partial^2 = 0$



$$\partial^2\langle x_0, x_1, x_2 \rangle = \langle x_1, x_2 \rangle - \langle x_0, x_2 \rangle + \langle x_0, x_1 \rangle =$$

$$\langle x_2 \rangle - \langle x_1 \rangle - (\langle x_2 \rangle - \langle x_0 \rangle) + \langle x_1 \rangle - \langle x_0 \rangle = 0$$

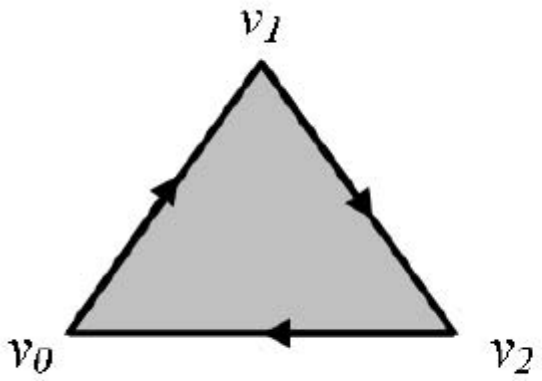
Simplicial complexes Why make these definitions?

… to define *boundaries*

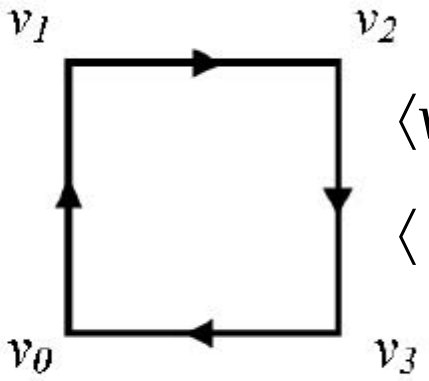
Any chain c_p with $\partial\, c_p = 0$ is defined to be a *cycle*

c_p is a *bounding cycle* if there is c_{p+1} with $\partial\, c_{p+1} = c_p$

$\langle v_1, v_2 \rangle - \langle v_0, v_2 \rangle + \langle v_0, v_1 \rangle$



(b) a bounding 1-cycle



$\langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle +$

$\langle v_2, v_3 \rangle + \langle v_3, v_0 \rangle$

(c) a non-bounding cycle

Simplicial complexes Why make these definitions?

… to define *boundaries*

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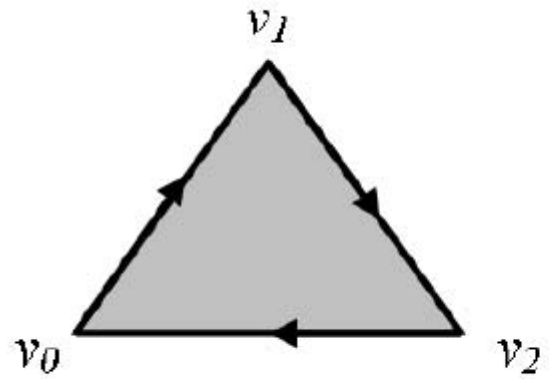
$\langle v_1, v_2 \rangle - \langle v_0, v_2 \rangle + \langle v_0, v_1 \rangle$

Non-bounding cycles characterise holes



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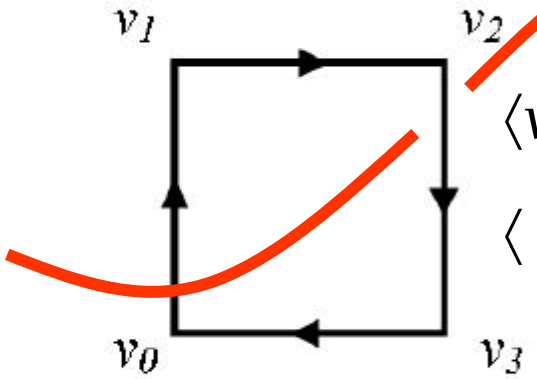


(c) a non-bounding cycle

$\langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle +$

$\langle v_2, v_3 \rangle + \langle v_3, v_0 \rangle$

Simplicial complexes Why make these definitions?

… to define *boundaries*

$\langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_0 \rangle$ is a non-bounding cycle

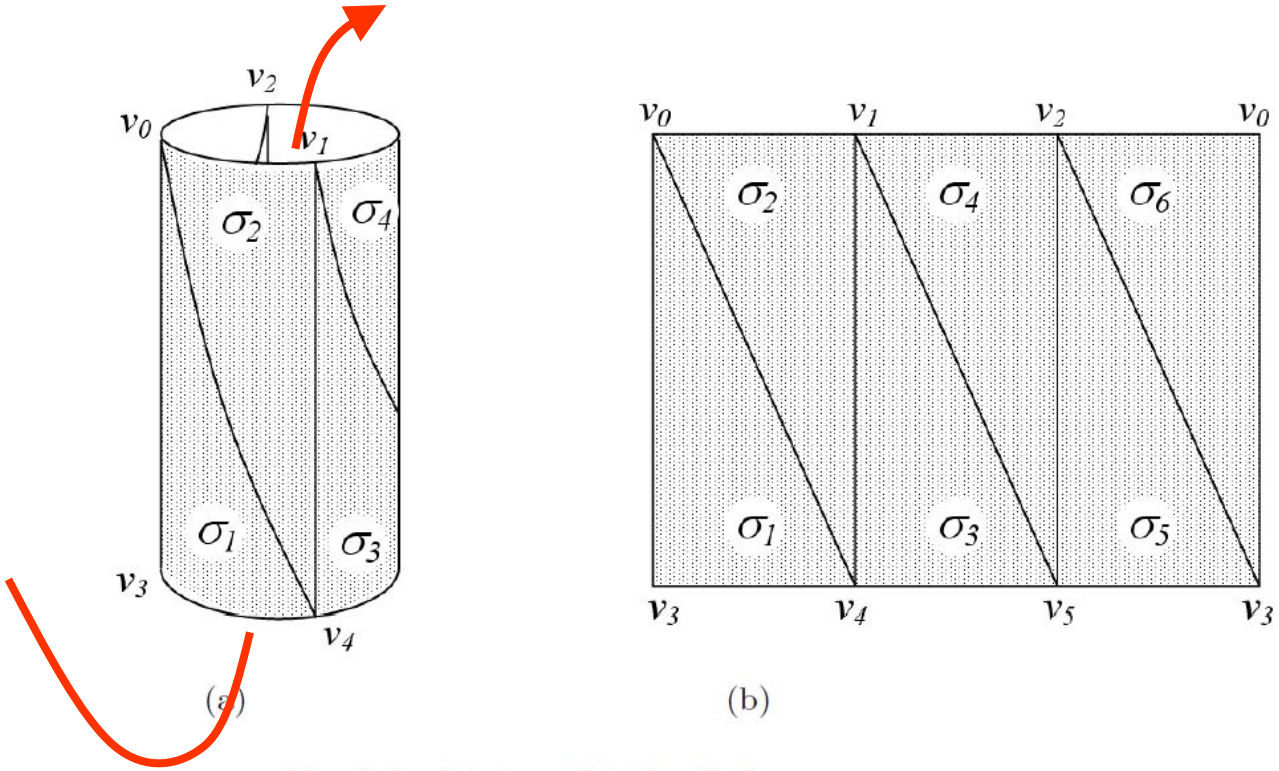


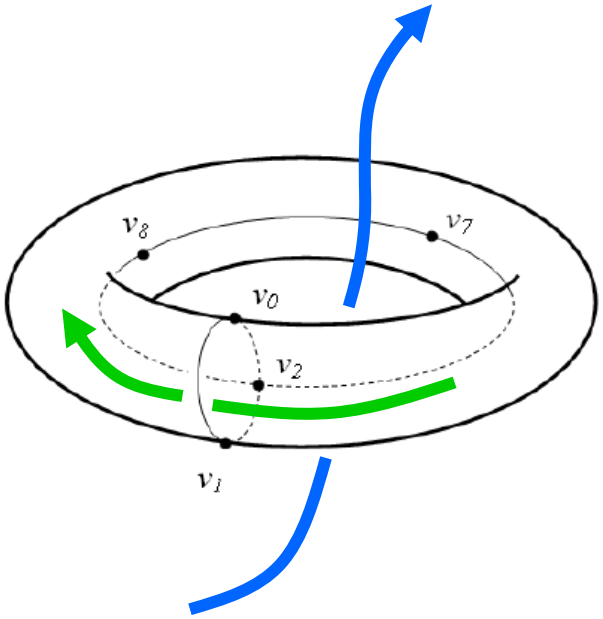
Fig. 2.8 A triangulated cylinder

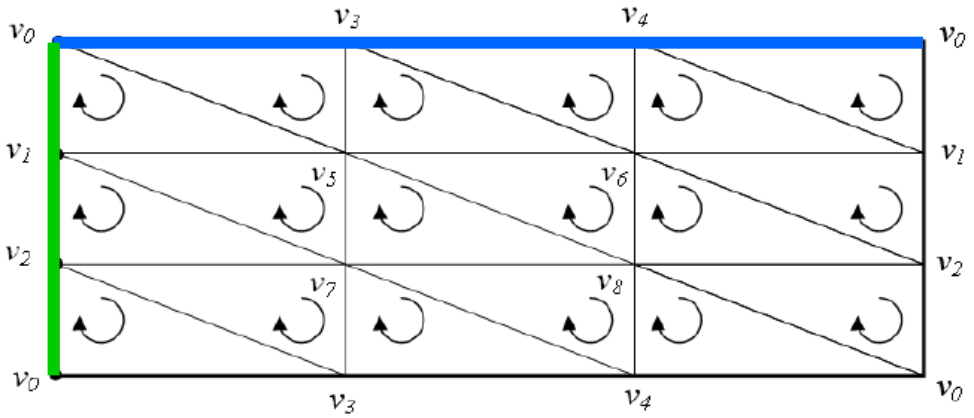
Simplicial complexes Why make these definitions?

… to define *boundaries*

$\langle v_{0,}, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle$ is a non-bounding cycle

$\langle v_{0,}, v_3\rangle + \langle v_3, v_4\rangle + \langle v_4, v_0\rangle$ is a non-bounding cycle



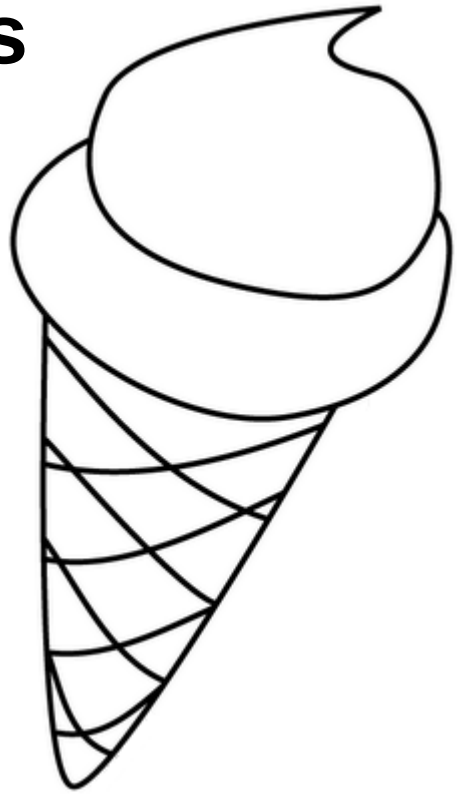


Part I : From networks to simplicial complexes

Part II: From simplicial complexes to hypernetworks

Part III: Issues in hypernetworks

From Networks to Hypernetworks

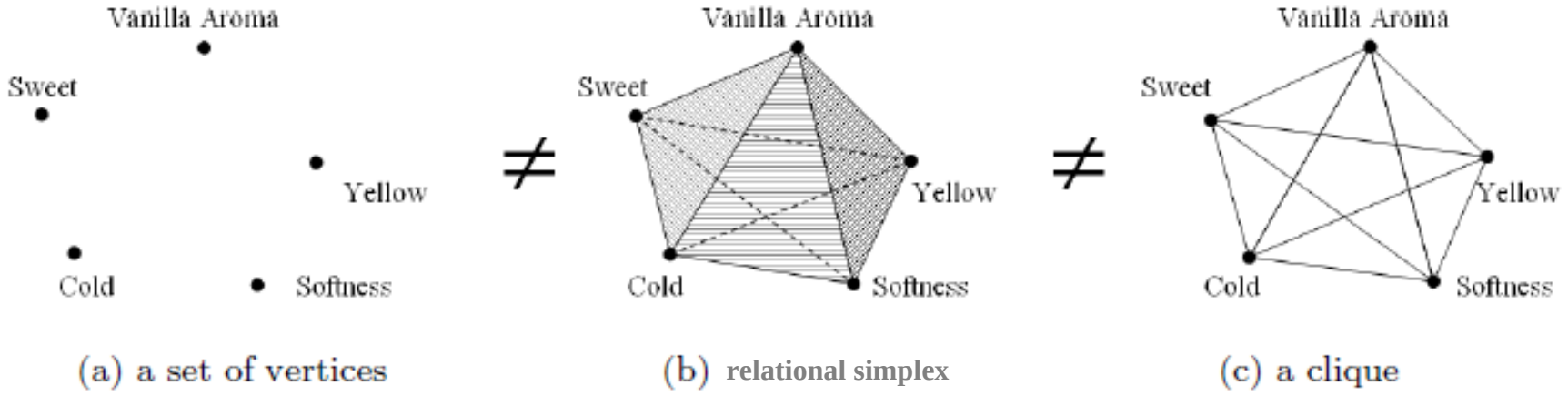


Gestalt Psychologist Katz:

Vanilla Ice Cream ≠ cold + yellow + soft + sweet + vanilla

it is a Gestalt – experienced as a whole

From Networks to Hypernetworks



(a) a set of vertices (b) relational simplex (c) a clique

Set of vertices ≠ relational simplex ≠ clique

From Networks to Hypernetworks

Gestalt Psychologist Katz:

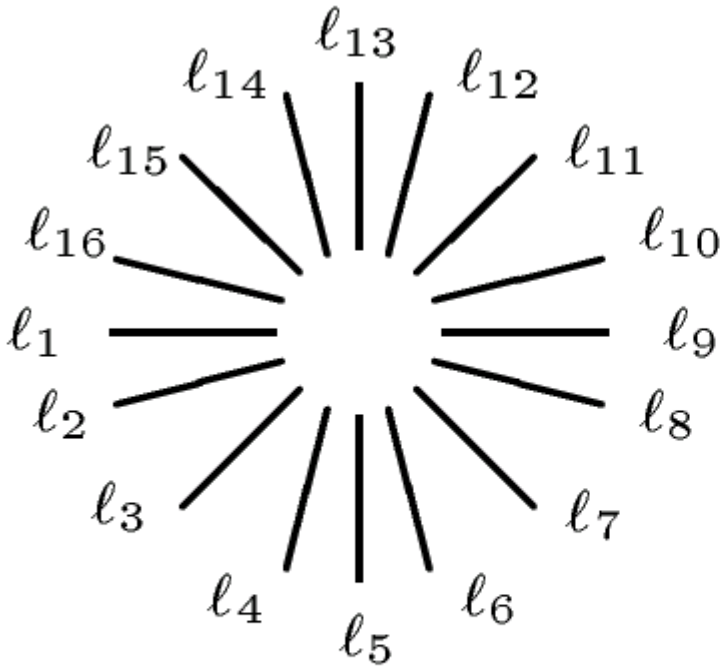
Vanilla Ice Cream ≠ cold + yellow + soft + sweet + vanilla

it is a Gestalt. **It is a *relational simplex***

$$\langle \textbf{cold, yellow, soft, sweet, vanilla; } R_{Vanilla\_Ice\_Cream} \rangle$$

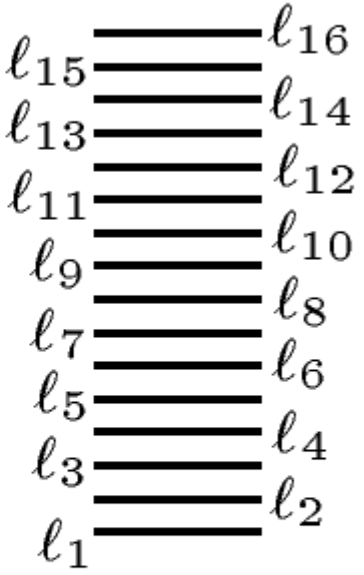
the relation is explicit

From Networks to Hypernetworks



(a) The sun illusion

$\sigma_1 = \langle \ell_1, ..., \ell_{16}; R_1 \rangle$



(b) the rectangle illusion

$\sigma_2 = \langle \ell_1, ..., \ell_{16}; R_2 \rangle$

Simplices by themselves are not rich enough to discriminate things

Here – same parts, different relation, different structure & emergence

We must have relational simplices

From Networks to Hypernetworks

Definition (to be revisited)

A ***hypernetwork*** is a set of relational simplices

e.g.

$\langle$ cold + yellow + soft + sweet + vanilla; $R_{Vanilla\_Ice\_Cream}\rangle$

The sun illusion
$\sigma_1 = \langle \ell_1, ..., \ell_{16}; R_1 \rangle$

the rectangle illusion
$\sigma_2 = \langle \ell_1, ..., \ell_{16}; R_2 \rangle$

Relational Simplices and Multilevel Systems

5.1 Systems of Systems of Systems

Most systems are characterised by having many subsystems and levels of description. They are made up of inextricably entangled multilevel social and physical subsystems with intra-level and inter-level bottom-up and top-top-down dynamics. They are *systems of systems*. In fact they are systems of systems of systems, and more generally multiple levels of systems of systems.

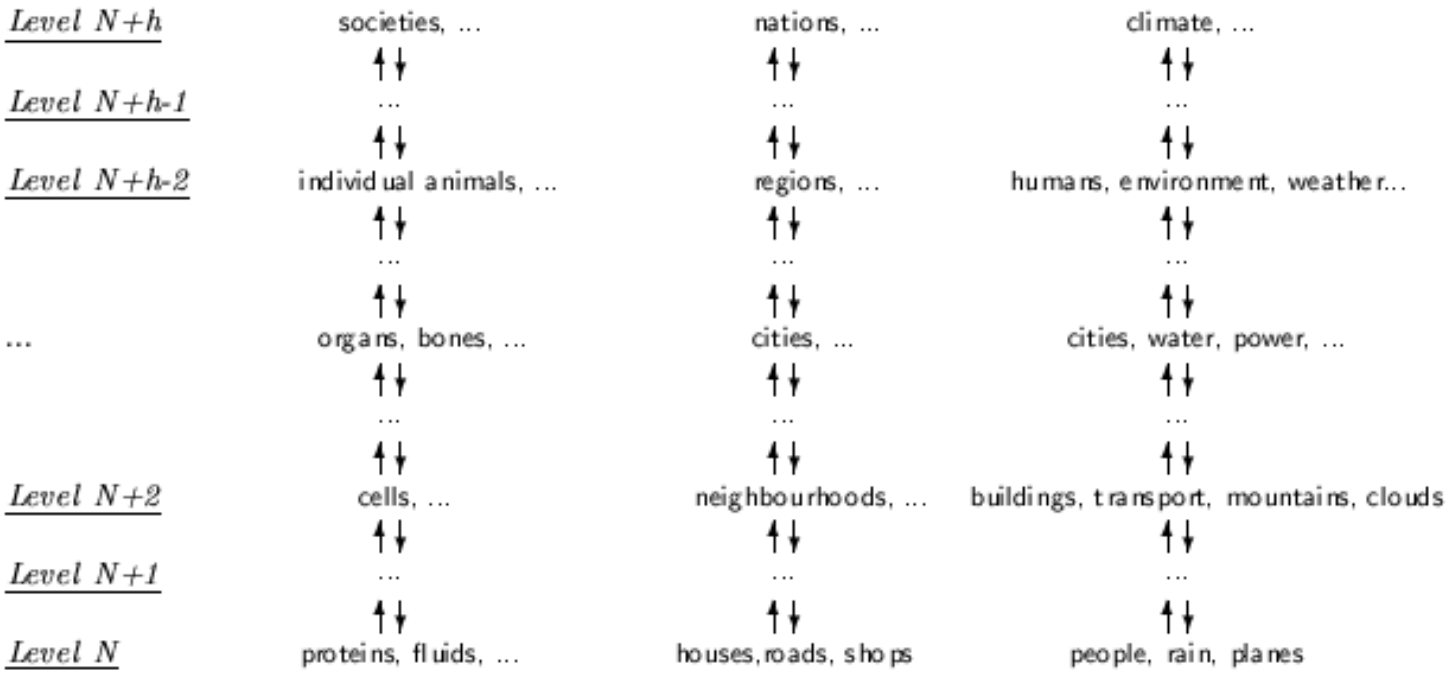


Fig. 5.1 Systems of systems of systems of systems ...

Multilevel Systems

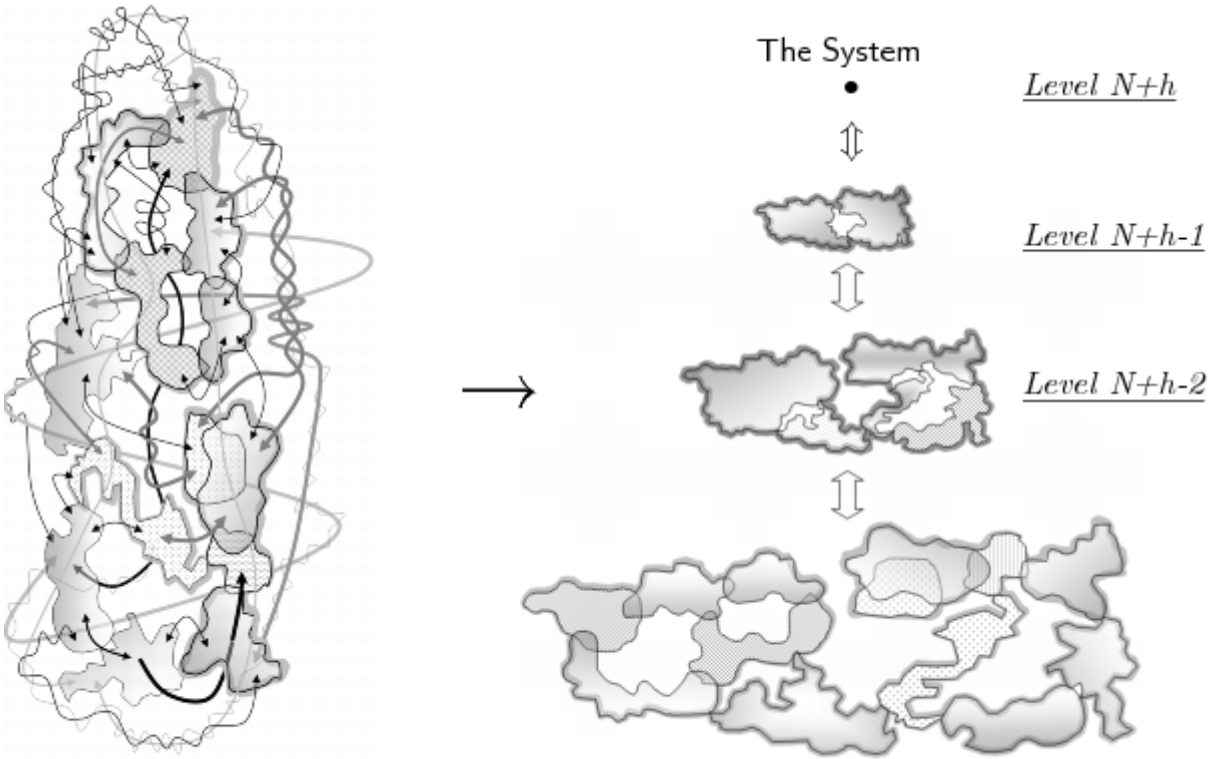


Fig. 5.2 Can highly entangled multilevel systems be resolved into well defined levels?

Multilevel Systems

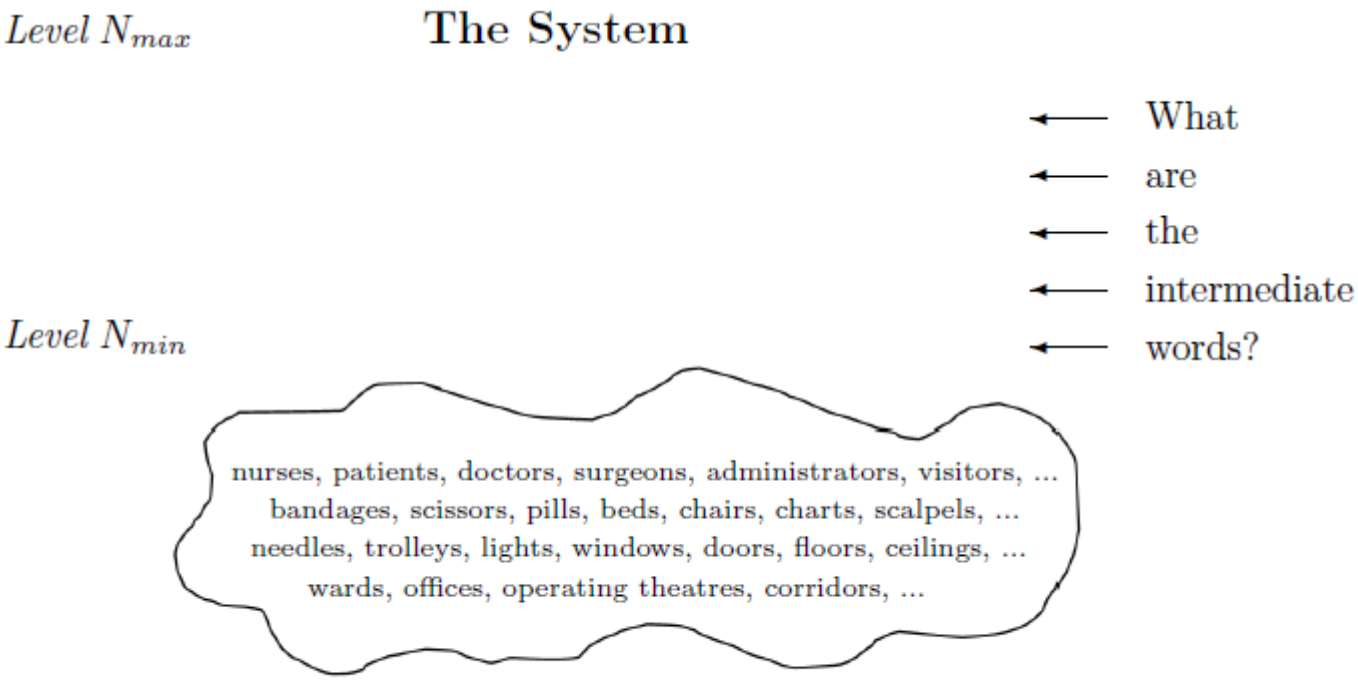


Fig. 5.3 The Intermediate Word Problem

Multilevel Systems

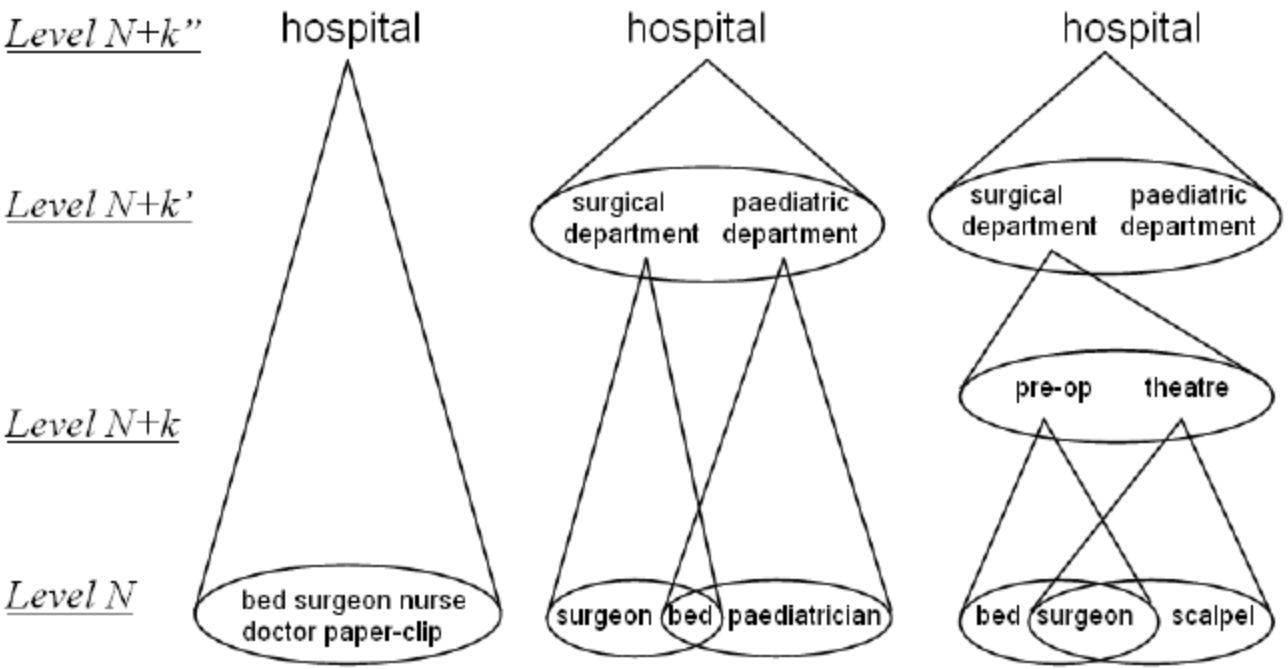
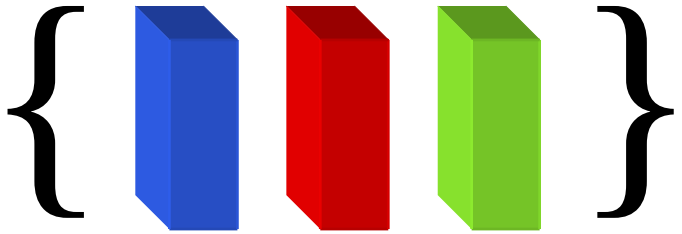


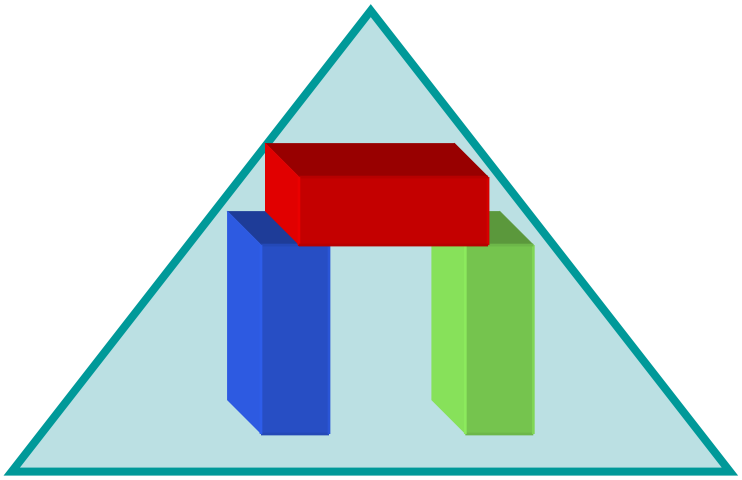
Fig. 5.4 Intermediate Words for a Hospital System

Formation of simplices ⇒ hierarchical structure

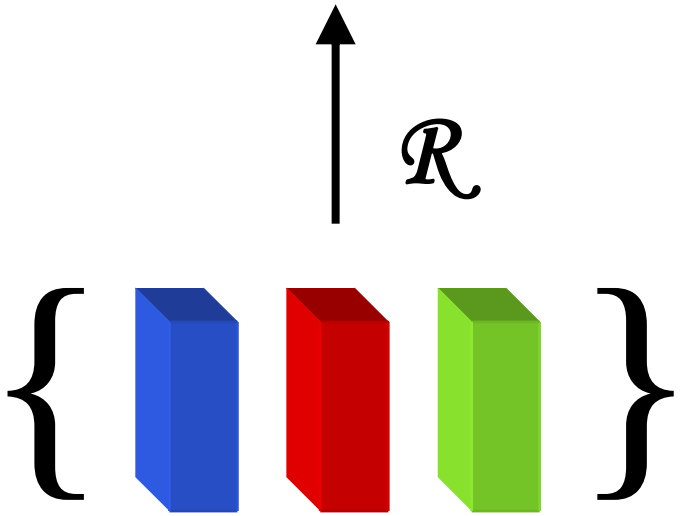
e.g. take a set of 3 blocks



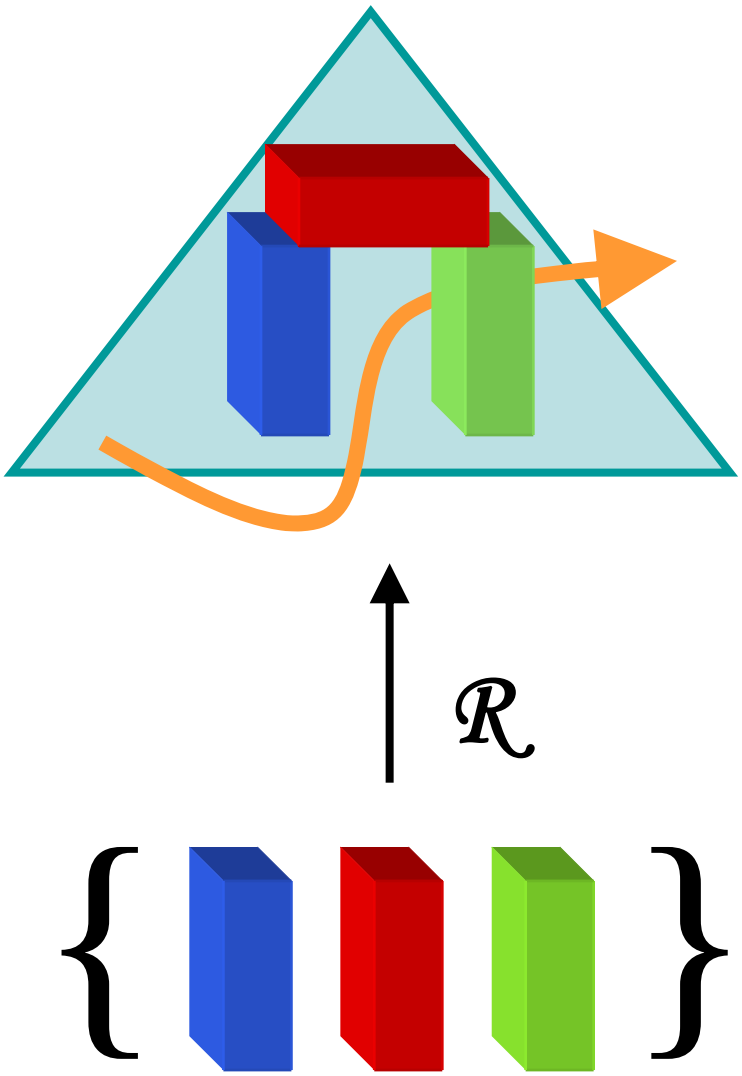
Formation of simplices ⇒ hierarchical structure



e.g. take a set of 3 blocks assembled by a 3-ary relation $\mathcal{R}$



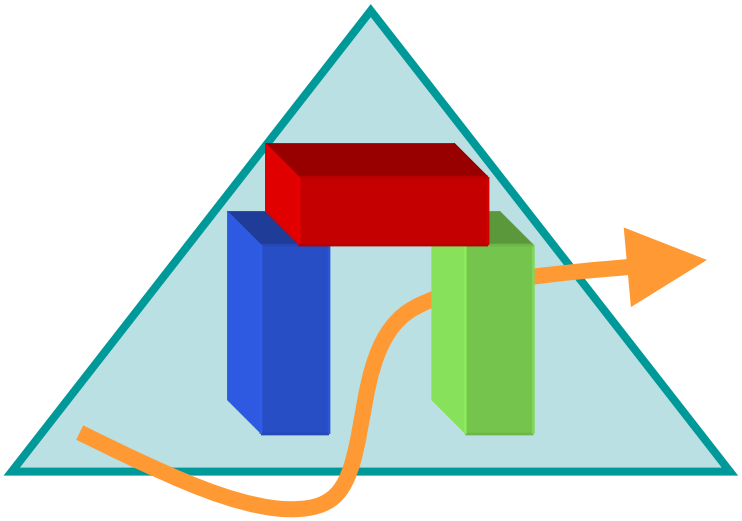
Formation of simplices ⇒ hierarchical structure



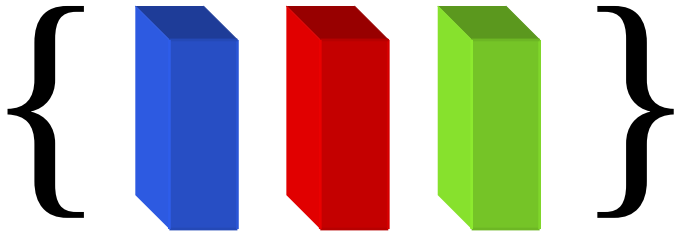
e.g. take a set of 3 blocks assembled by a 3-ary relation $\mathcal{R}$

The structure has an emergent property

Formation of simplices ⇒ hierarchical structure



$\mathcal{R}$

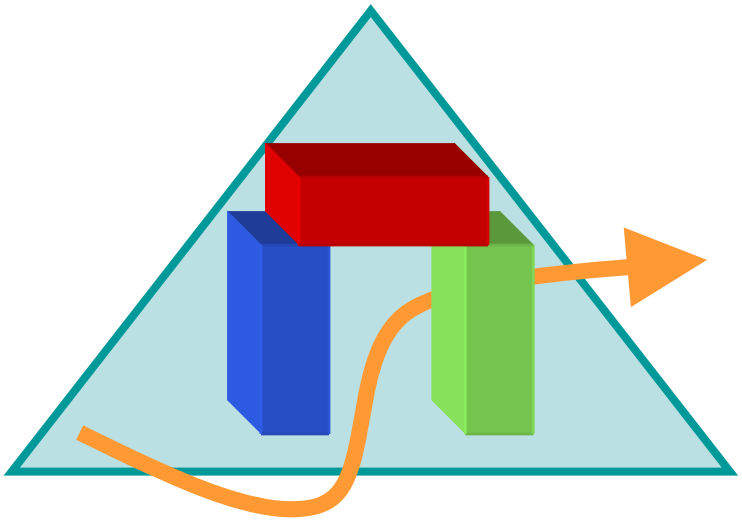


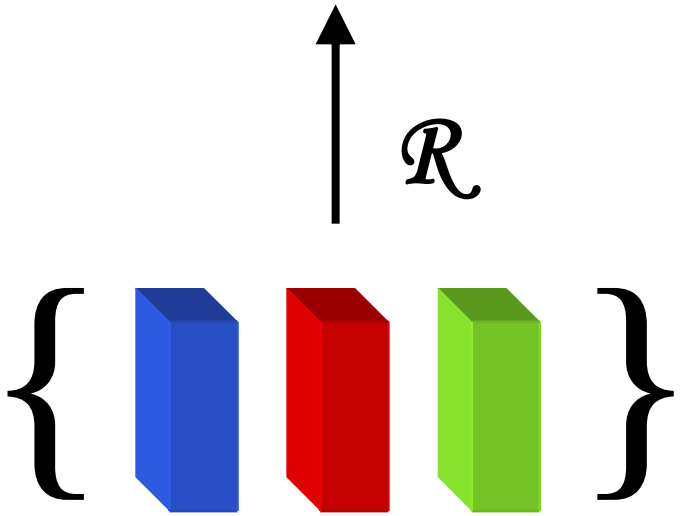
n-ary relation assembles elements into named structures at a higher level

Level N+1

Level N

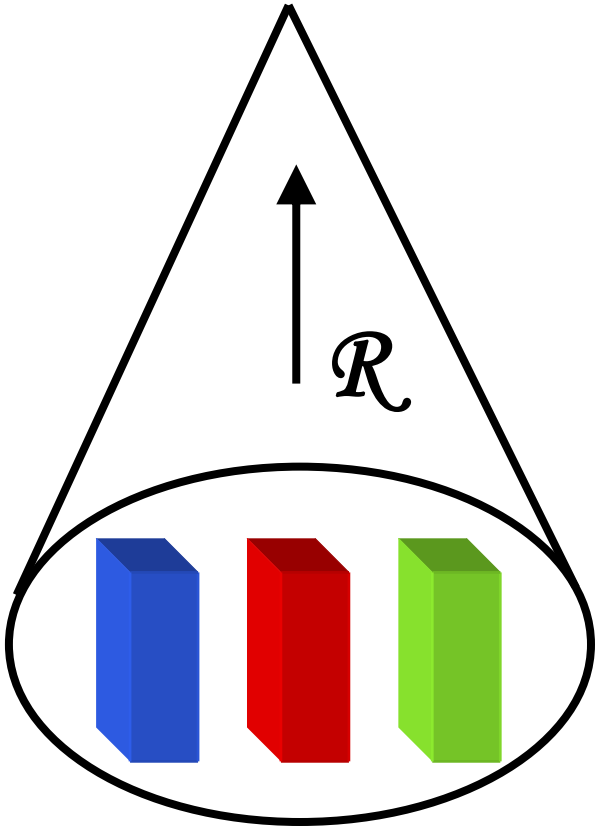
Formation of simplices ⇒ hierarchical structure





n-ary relation assembles elements into named structures at a higher level

Arch



5.4 Relational Simplices and The Axiom of Aggregation

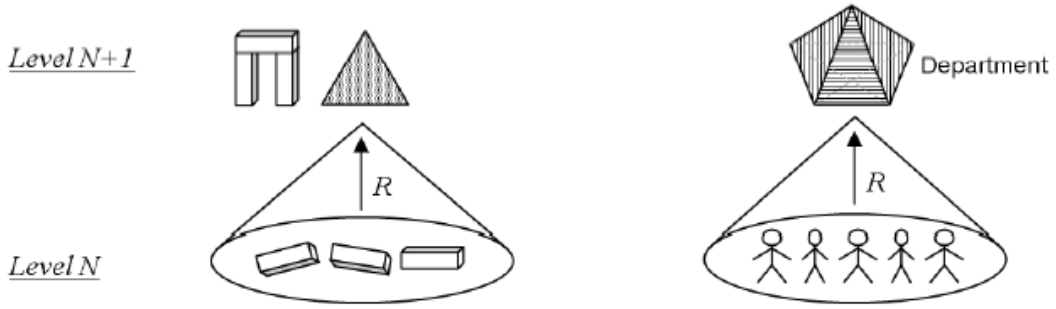


Fig. 5.5 Multilevel Aggregation

The goal of finding a well defined way of representing multilevel systems requires a way of distinguishing between levels. Figure 5.5 shows the assembly of elements at *Level N* to structures at a higher level denoted *Level N+1*. This notation is used to make it clear that the integer values of levels are not absolute and if necessary *Level* $N+k$ may be relabeled as *Level* $N+k'$. The notation $Level(x) = N+k$ will mean that x exists in the representation of the system at *Level N+k*.

The Axioms of Aggregation: Let $\sigma = \langle x_0, ..., x_n; R\rangle$ be a relational simplex.

(i) Antisymmetry: $Level(x_i) < Level(\sigma)$, for $i = 0, ..., n$.

(ii) Transitivity: $Level(x) < Level(y)$ and $Level(y) < Level(z)$ imply $Level(x) < Level(z)$.

This axiom asserts that elements exist at a lower level of aggregation to the structures they form. It is a variant on the *mereological* axiom that if x is a 'proper part' of y, then y is not a proper part of x.

AND and OR aggregations in multilevel systems

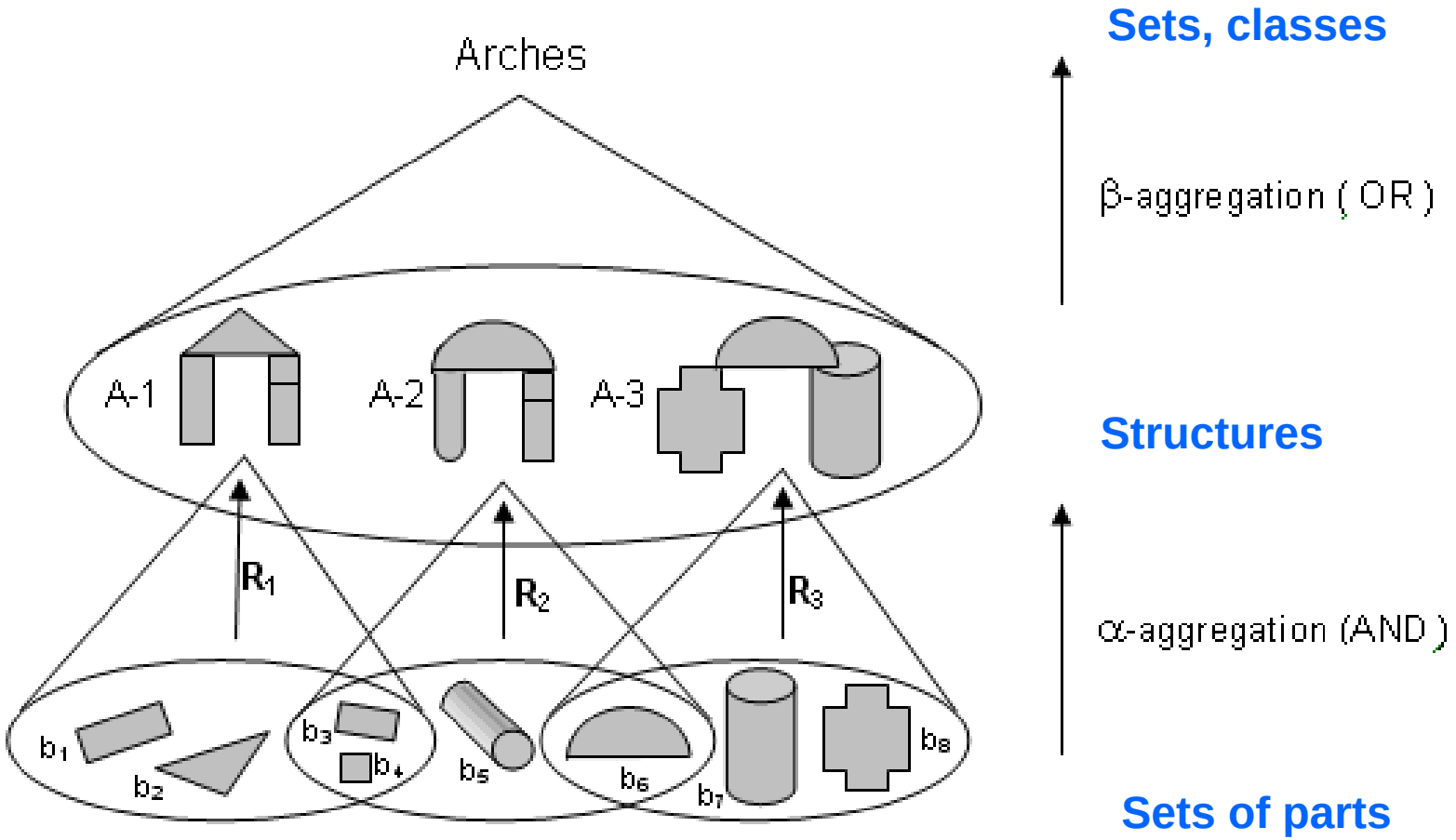


Figure 8. Two different types of multilevel aggregation

Conventional classification trees don't have alpha aggregations

Observing multilevel systems of systems of systems

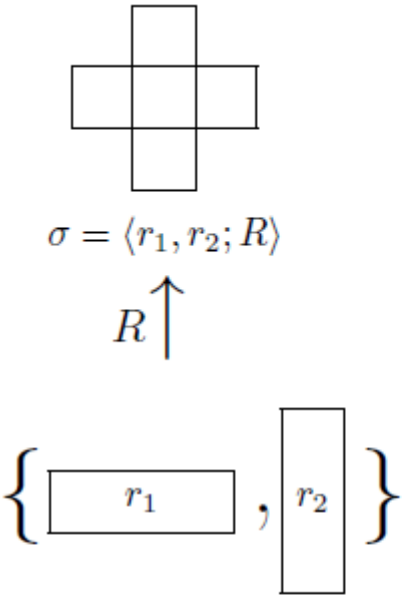
Hypothesis 1

When we look at systems we see the whole & the parts

Hypothesis 2

Our brains create new tructures

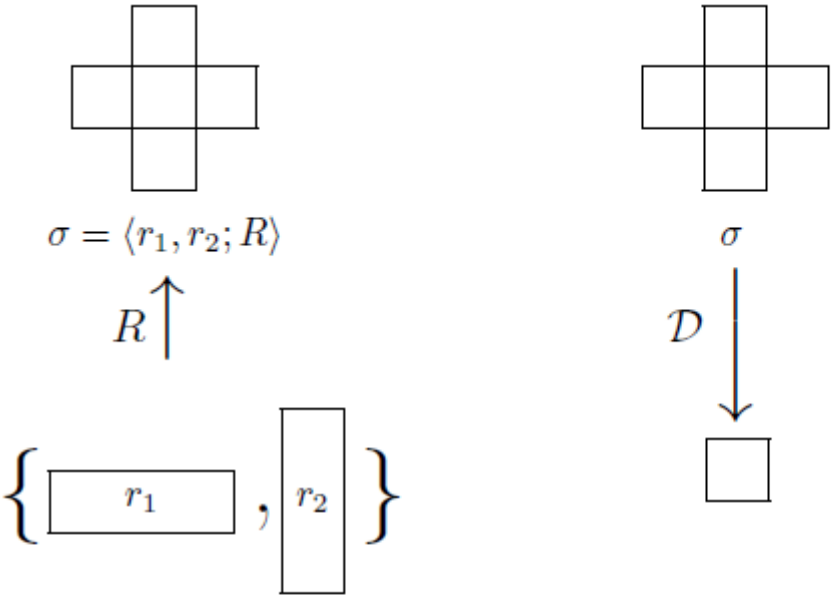
Aggregation – deconstruction downward dynamics in representing systems



Level N+1

Level N

Aggregation – deconstruction downward dynamics in representing systems

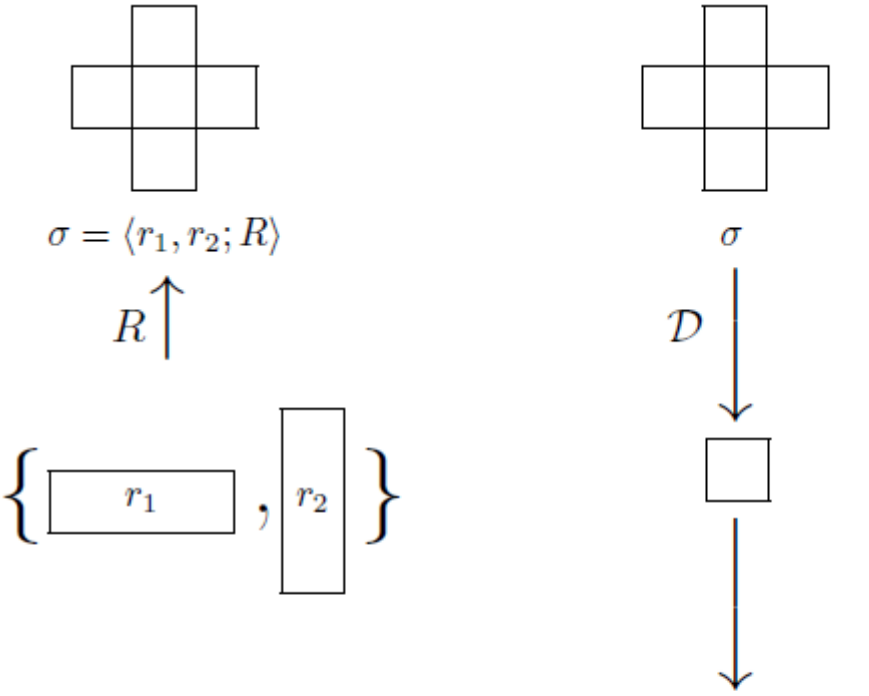


Level N+1

Level N

Create a new object at Level N !

Aggregation – deconstruction downward dynamics in representing systems



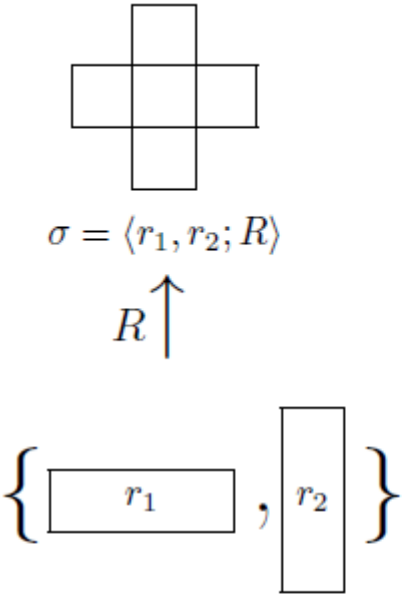
Level N+1

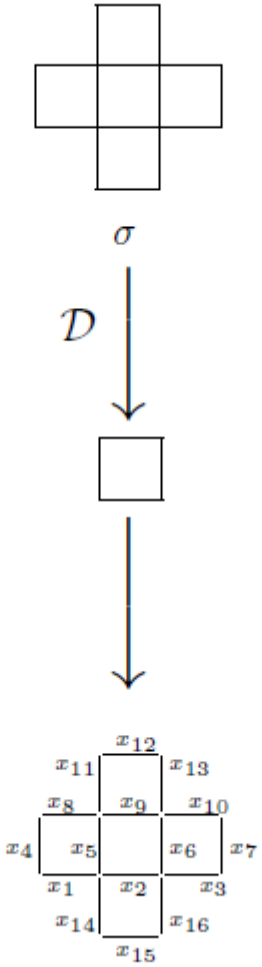
Level N

Create a new level - Level N-1 !

And new objects at this level

Aggregation – deconstruction downward dynamics in representing systems

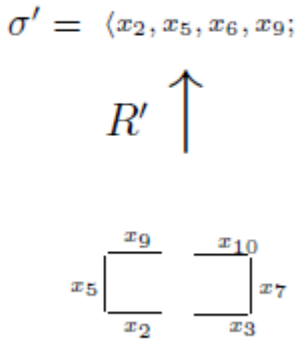




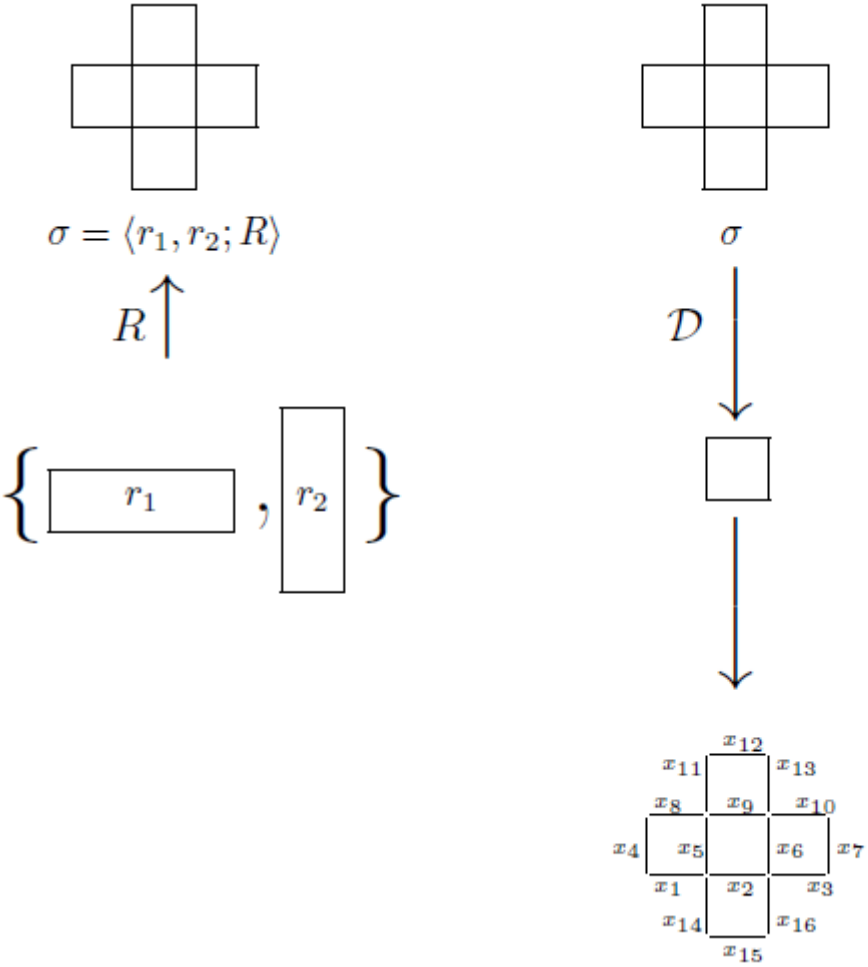
Level N+1

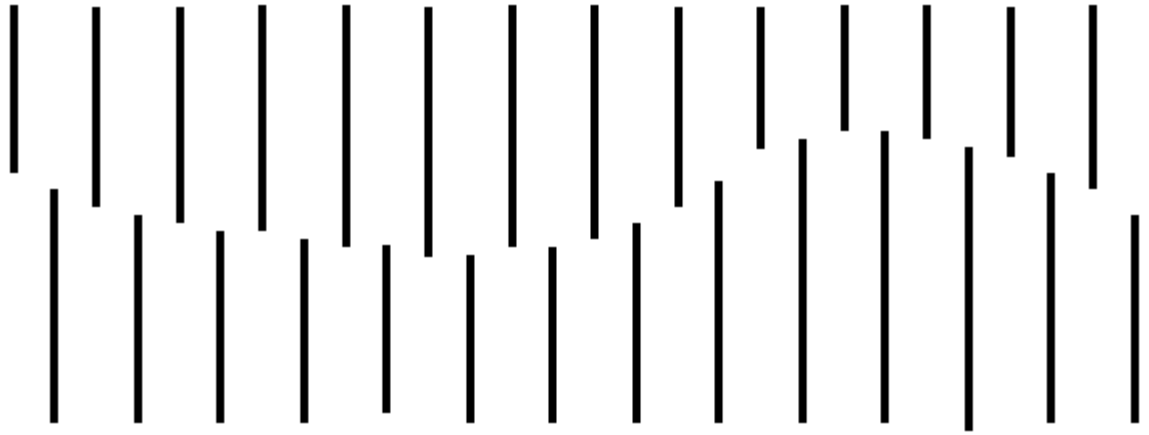
Create a new object at Level N

Level N



Aggregation – deconstruction downward dynamics in representing systems

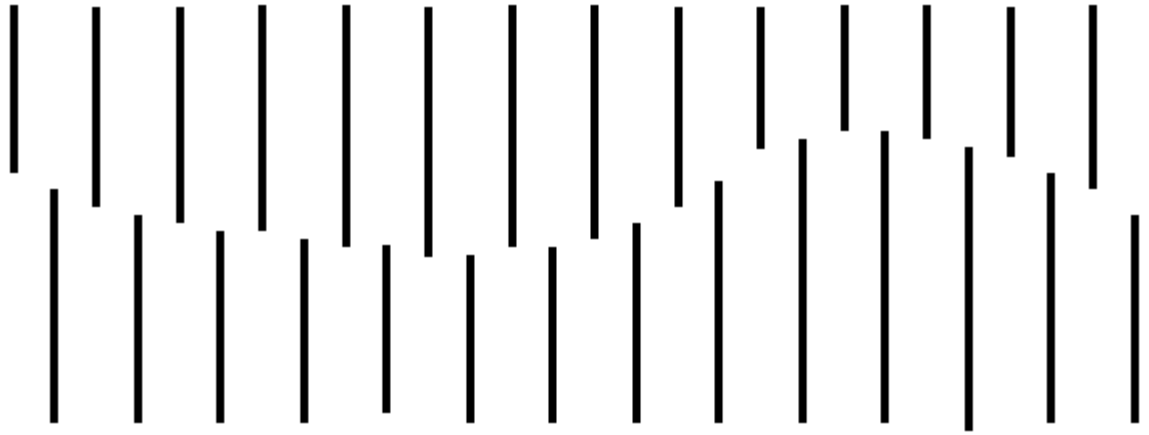




(c) the virtual contour illusion

$\sigma_3 = \langle \ell_1, ..., \ell_{28}; R_3 \rangle$

Q: Does the virtual contour exist?



(c) the virtual contour illusion

$\sigma_3 = \langle \ell_1, ..., \ell_{28}; R_3 \rangle$

Q: Does the virtual contour exist?

A: Yes. Our brains make kinds of virtual stuff out of everything.

Computers can (and must!) also make these virtual structures for representing multilevel systems.

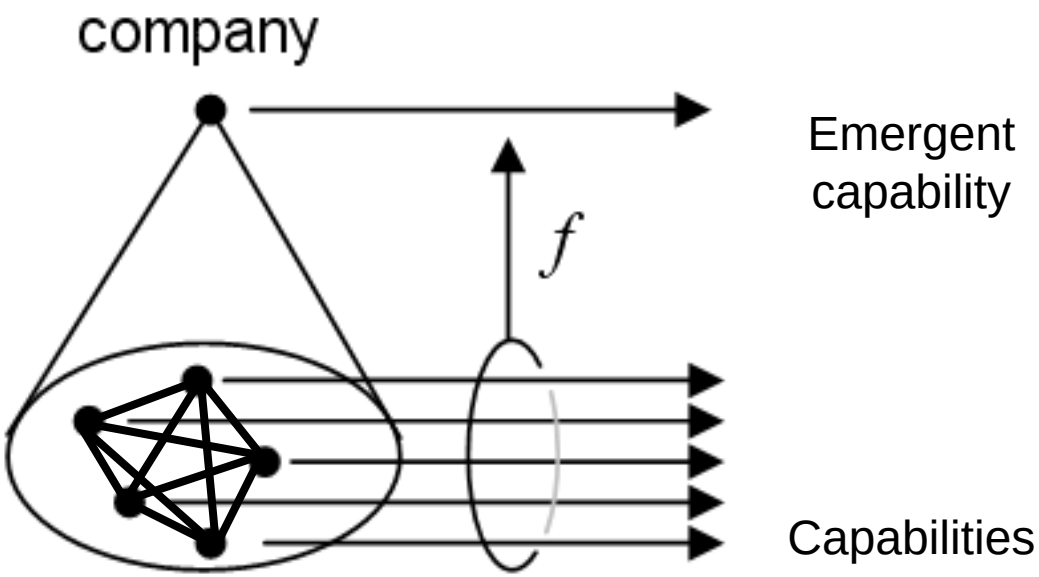
Backcloth and traffic

Relational simplices support patterns of numbers across their faces representing dynamical aspects of the systems.

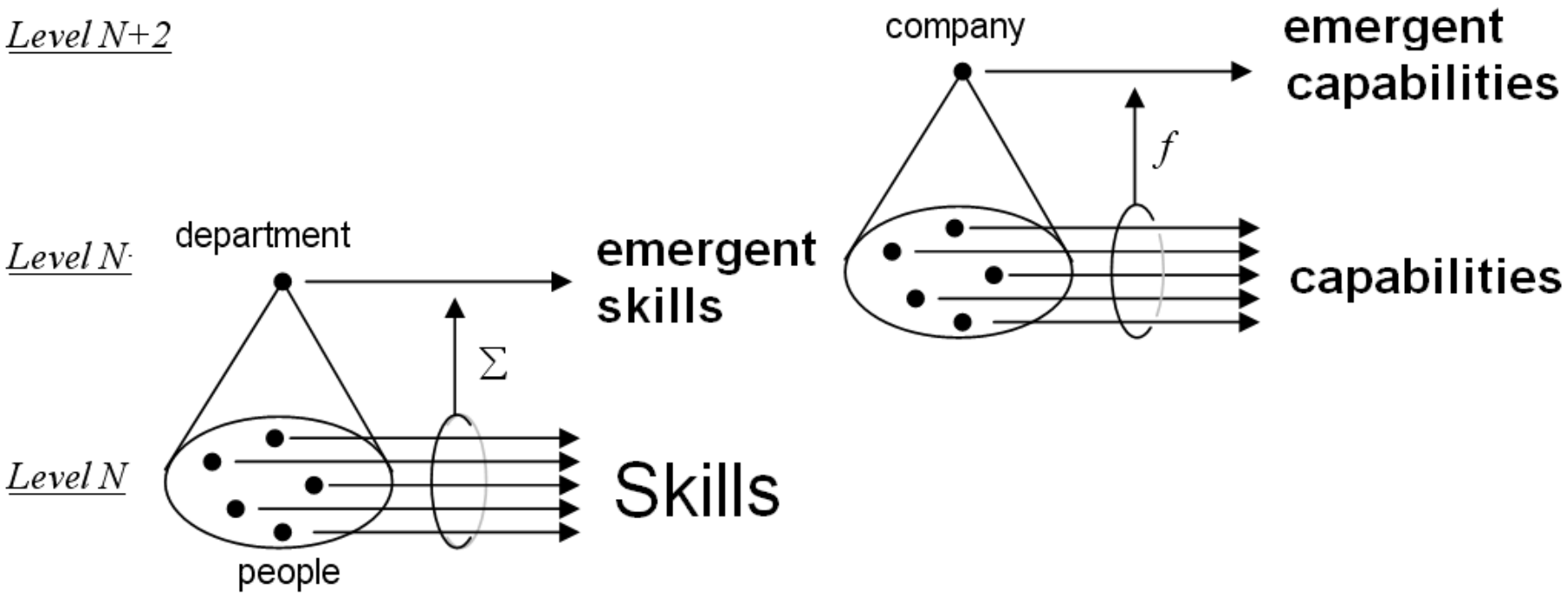
The simplices form a backcloth for the more dynamic traffic

… but there are also backcloth dynamics as relational simplices are formed.

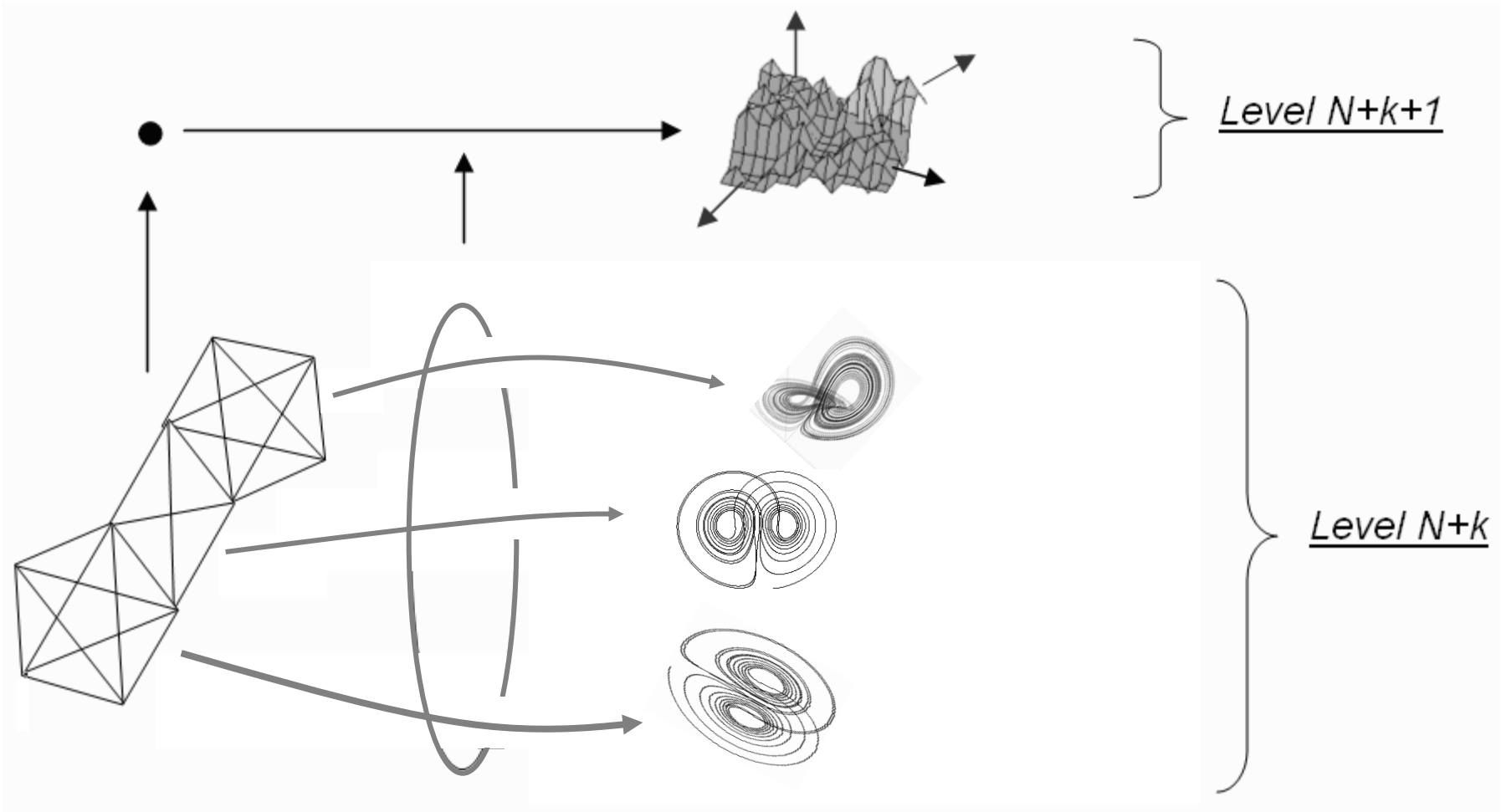
Multilevel patterns of numbers on the structure



Multilevel patterns of numbers on the structure



Dynamics on the hypernetwork backcloth



System dynamics as traffic on a fixed multilevel backcloth

Dynamics

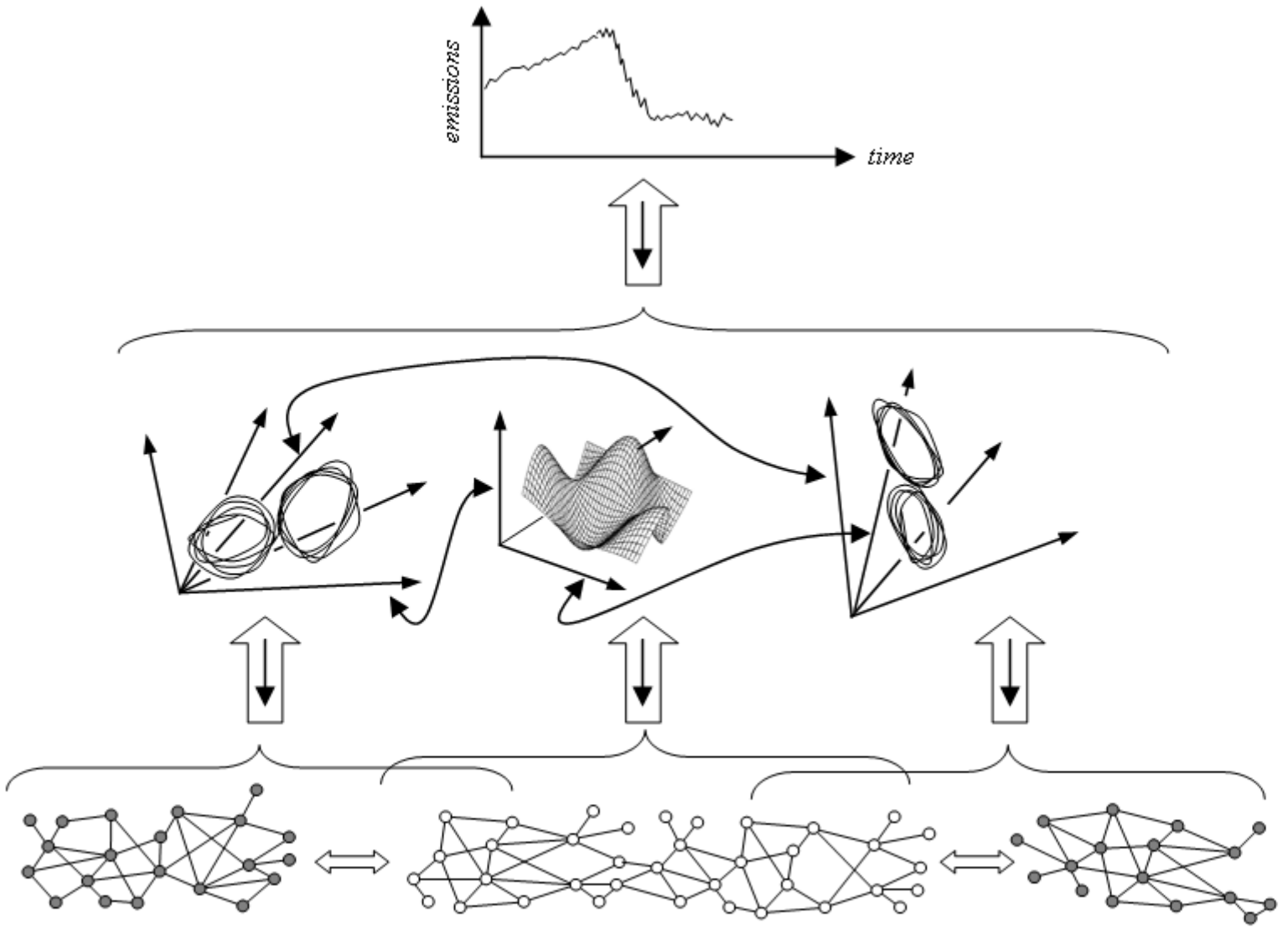


Figure 2. discrete micro-dynamic, semi-continuous meso-dynamics and continuous macro-dynamic

Backcloth dynamics: System time and System Events

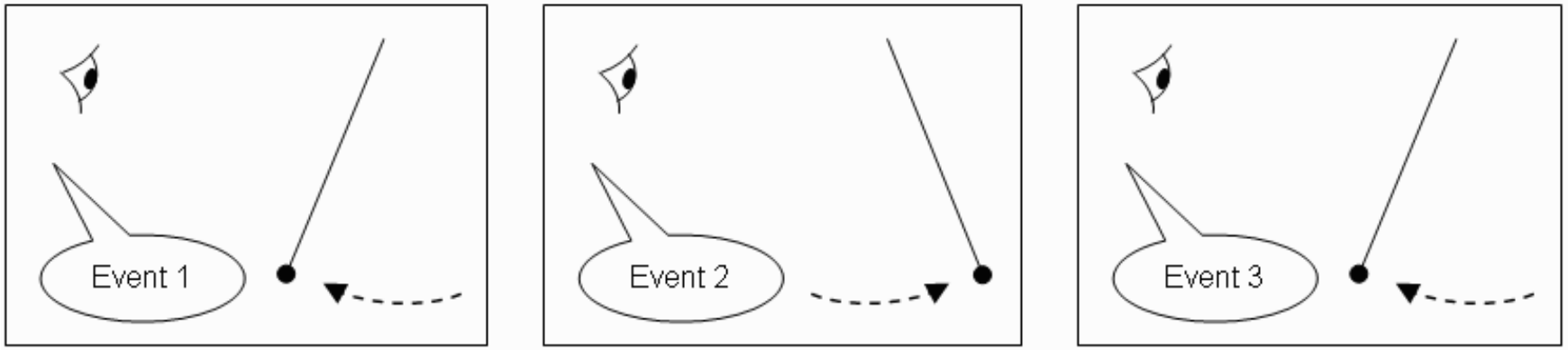


Figure 18: Pendulum events used to measure clock time

Backcloth dynamics: System time and System Events

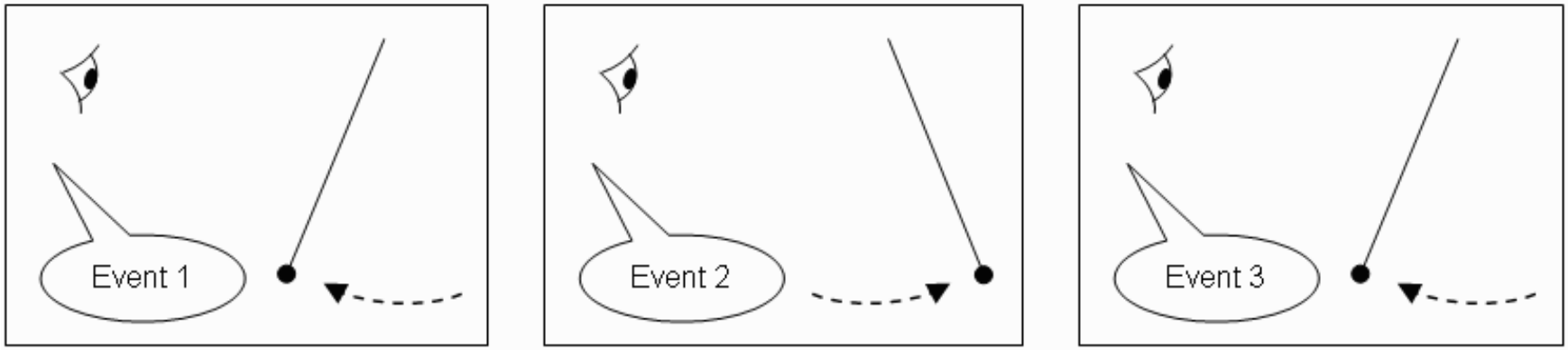


Figure 18: Pendulum events used to measure clock time

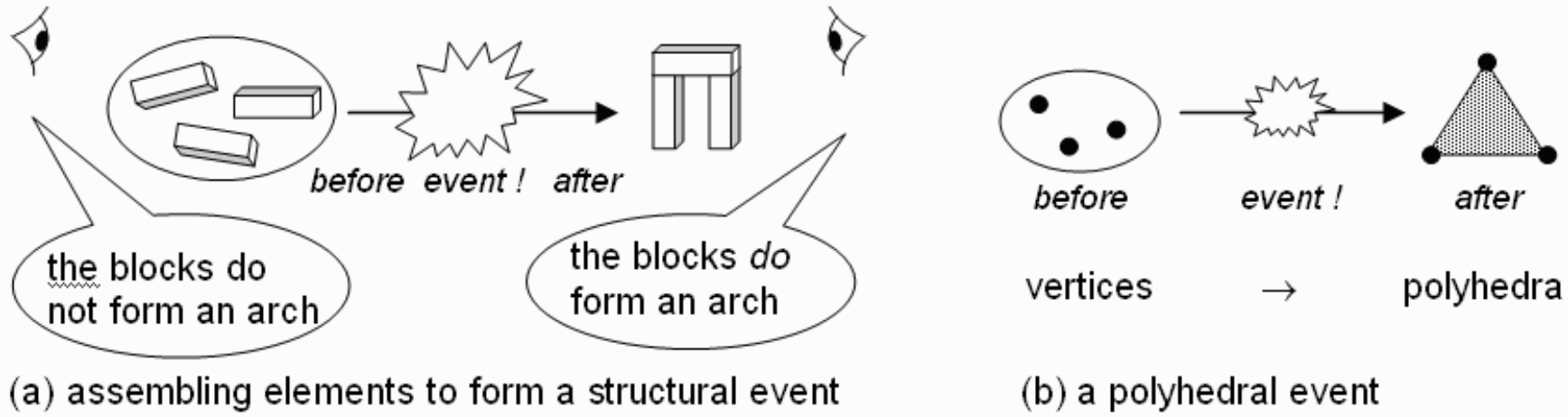


Figure 19. The formation of polyhedral structure marks system events

Backcloth dynamics: System time and System Events

N-ary relations have their own dynamics

$$R_t : \{\ a, b, c, d, \ldots.\ \} \rightarrow \langle\ a, b, c, d, \ldots; R_{t+k}\ \rangle$$

For the relation to act takes elapsed time.

In natural systems (biology) these dynamics may involve traffic

In artificial systems these dynamics are *designed* & managed.

Backcloth dynamics: System time and System Events

N-ary relations have their own dynamics

$$R_t : \{\ a, b, c, d, \ldots.\ \} \rightarrow \langle\ a, b, c, d, \ldots; R_{t+k}\ \rangle$$

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Backcloth dynamics: System time and System Events

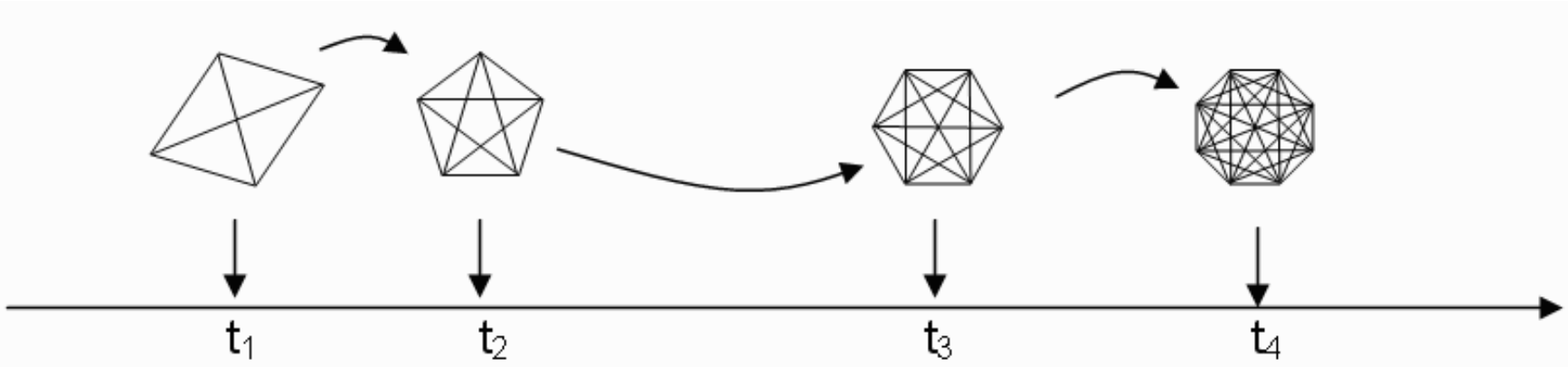


Figure 21. System event dynamics form trajectories in a non-linear way in clock time

Planning involves changing *relations*

Backcloth dynamics: System time and System Events

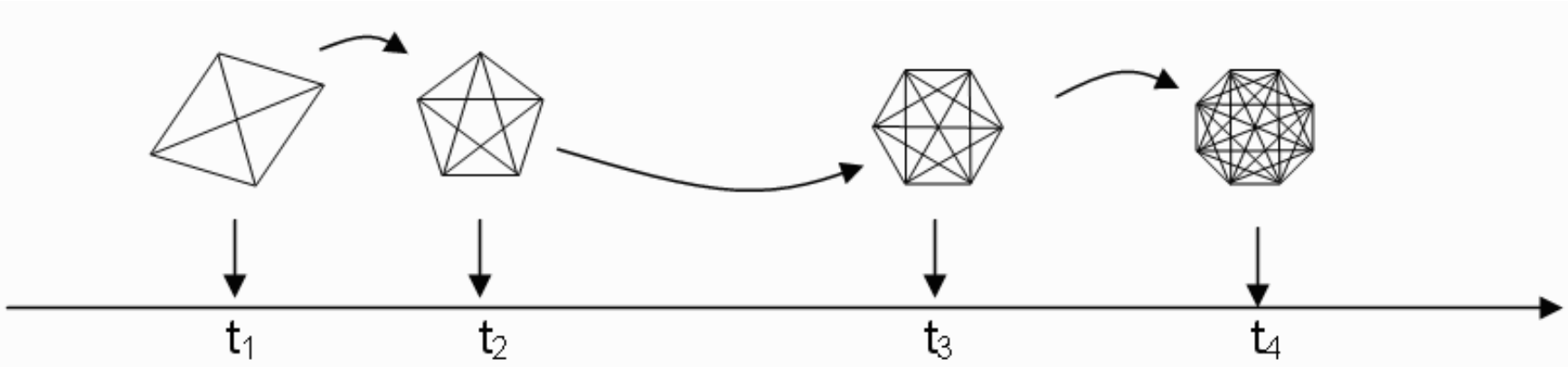


Figure 21. System event dynamics form trajectories in a non-linear way in clock time

System dynamics involves changing *relations*

… trajectories of multidimensional events

Part I : From networks to simplicial complexes

Part II: From simplicial complexes to hypernetworks

Part III: Issues in hypernetworks

Simplicial complexes to hypernetworks

A hypernetwork is a ***set*** of relational simplices ?

A hypernetwork is a ***chain*** algebra of relational simplices?

In chain hypernetworks, what do orientation & minus mean?

Simplicial complexes to hypernetworks

f rather than ∂ ?

$$f : \langle v_0, v_1, \ldots v_p \rangle \rightarrow \{ \langle v_0, \ldots, \widehat{v_i}, \ldots v_p \rangle \mid i = 0, \ldots, p \}$$

$$\widehat{f}^p : \langle v_0, v_1, \ldots v_p \rangle \rightarrow \{ v_0, v_1, \ldots v_p \}$$

Or

$$f : \langle v_0, v_1, \ldots v_p \rangle \rightarrow \Sigma_i \langle v_0, \ldots, \widehat{v_i}, \ldots v_p \rangle$$

$$\widehat{f}^p : \langle v_0, v_1, \ldots v_p \rangle \rightarrow \langle v_0 \rangle + \ldots + \langle v_p \rangle$$

Simplicial complexes to hypernetworks

Fundamental question

$$\langle x_0, x_1, x_2, \ldots;\ R_1 \rangle \cap \langle y_0, y_1, y_2, \ldots;\ R_1 \rangle = \langle z_0, z_1, z_2, \ldots.;\ R_1 \ ? \ R_2 \rangle$$

where $\{x_0, x_1, x_2, \ldots\} \cap \langle y_0, y_1, y_2, \ldots\} = \{z_0, z_1, z_2, \ldots\}$

Simplicial complexes to hypernetworks

Fundamental question

$$\langle x_0, x_1, x_2, \ldots;\ R_1 \rangle \cap \langle y_0, y_1, y_2, \ldots;\ R_1 \rangle = \langle z_0, z_1, z_2, \ldots.;\ R_1 \,?\, R_2 \rangle$$

where $\{x_0, x_1, x_2, \ldots\} \cap \langle y_0, y_1, y_2, \ldots\} = \{z_0, z_1, z_2, \ldots\}$

Relational algebra, *e.g.*

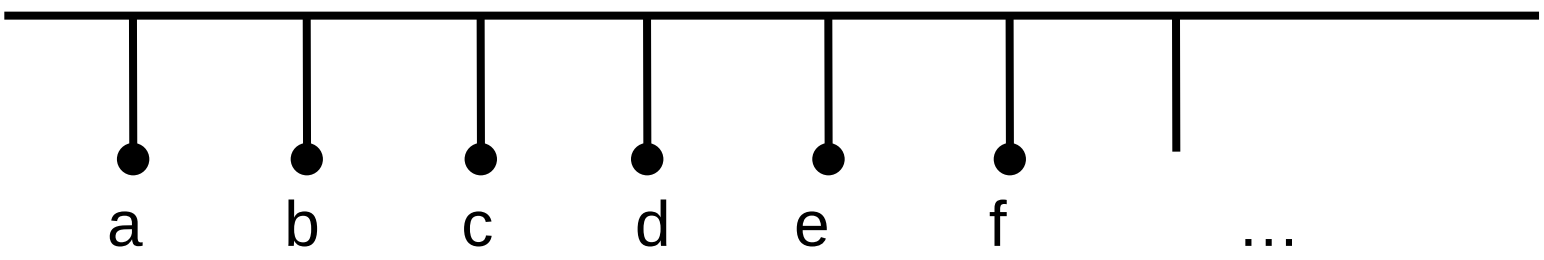
$$\langle\ v_0, v_1, \ldots.\ v_p\ ; R_1 \wedge R_2 \rangle \rightarrow\ \langle\ v_0, \ldots,\ v_p; R_1 \rangle + \langle\ v_0, \ldots,\ v_p; R_1 \rangle$$

Simplicial complexes to hypernetworks

In chain hypernetworks, what do orientation & minus mean?

Perhaps ‘plus’ means ‘add to the database’ ?

(a + b + c + d + e + f) + g

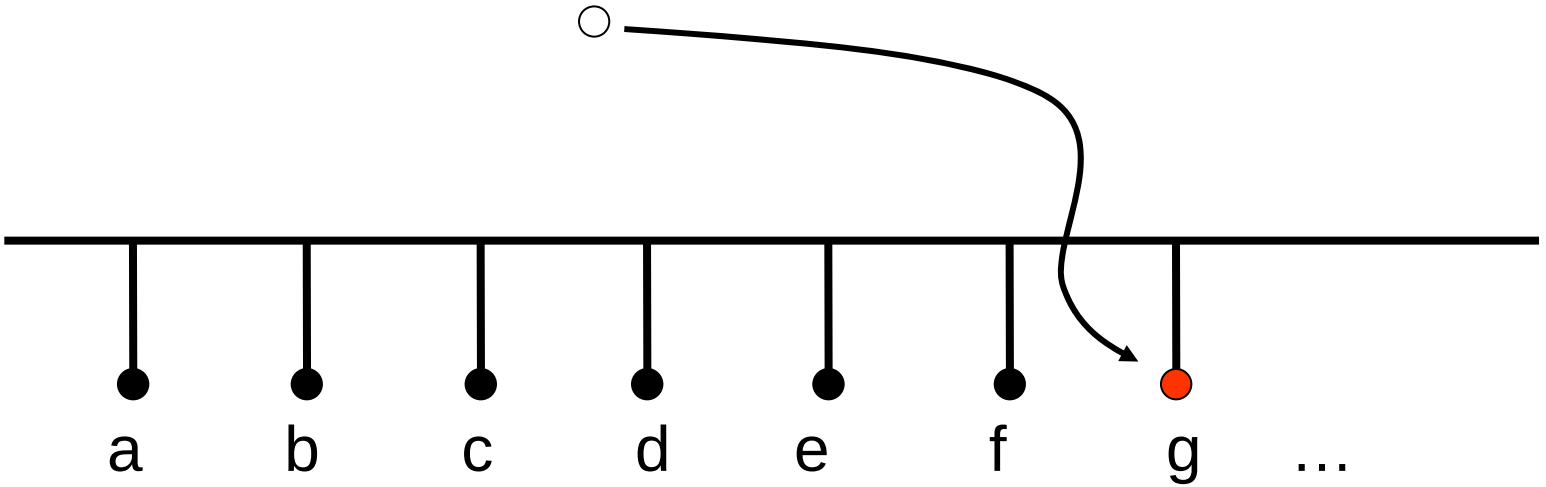


Simplicial complexes to hypernetworks

In chain hypernetworks, what do orientation & minus mean?

Perhaps ‘plus’ means ‘add to the database’ ?

(a + b + c + d + e + f) + g = a + b + c + d + e + f + g ?

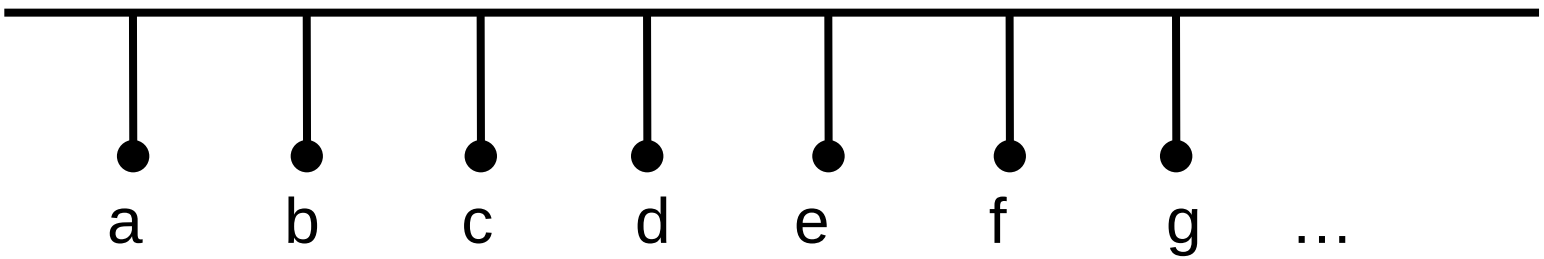


Simplicial complexes to hypernetworks

In chain hypernetworks, what do orientation & minus mean?

Perhaps ‘minus’ means ‘remove from the database’ ?

(a + b + c + d + e + f + g) - f =



Simplicial complexes to hypernetworks

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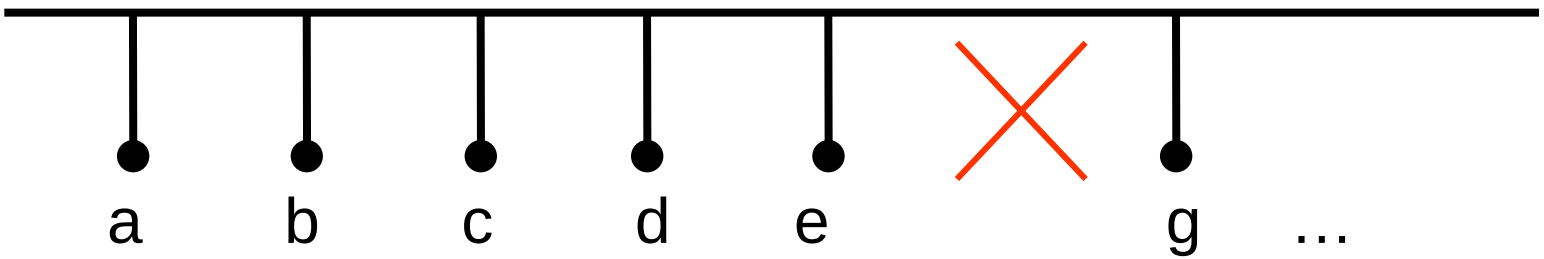

Simplicial complexes to hypernetworks

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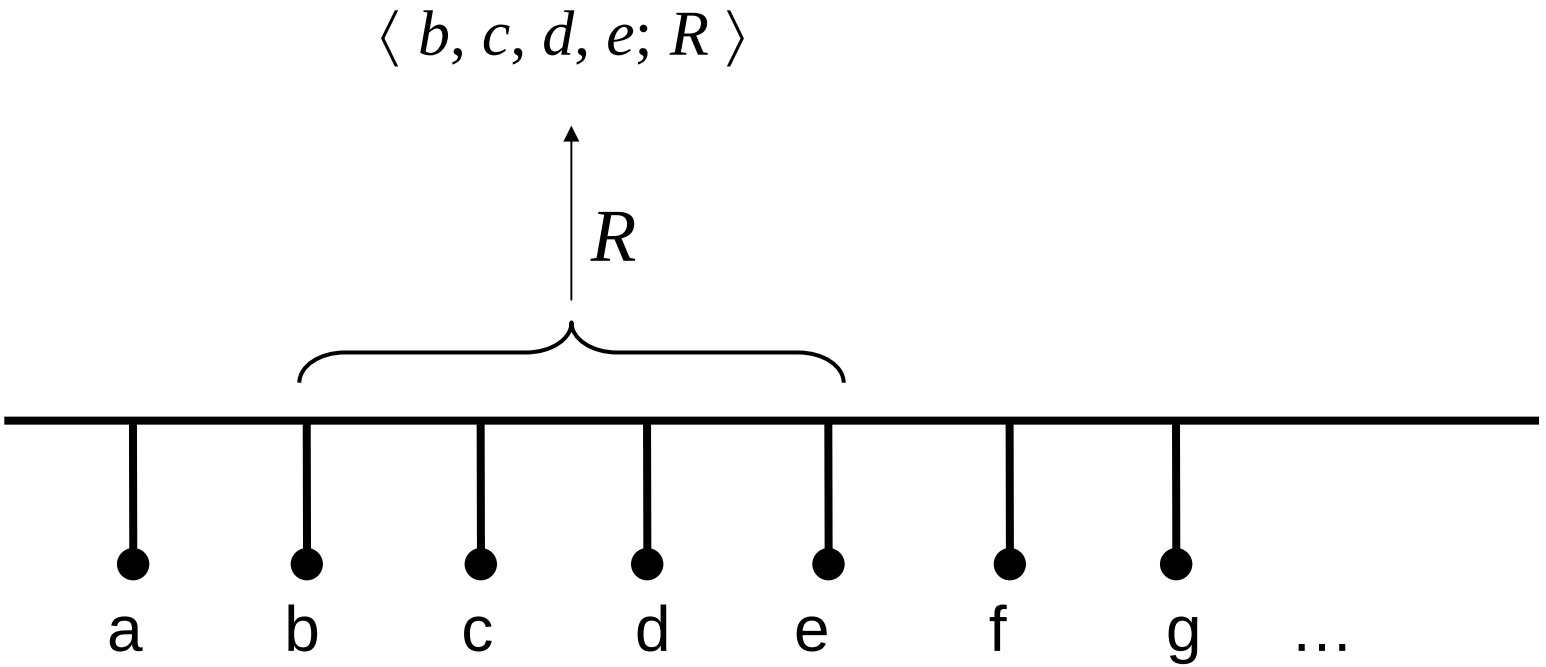
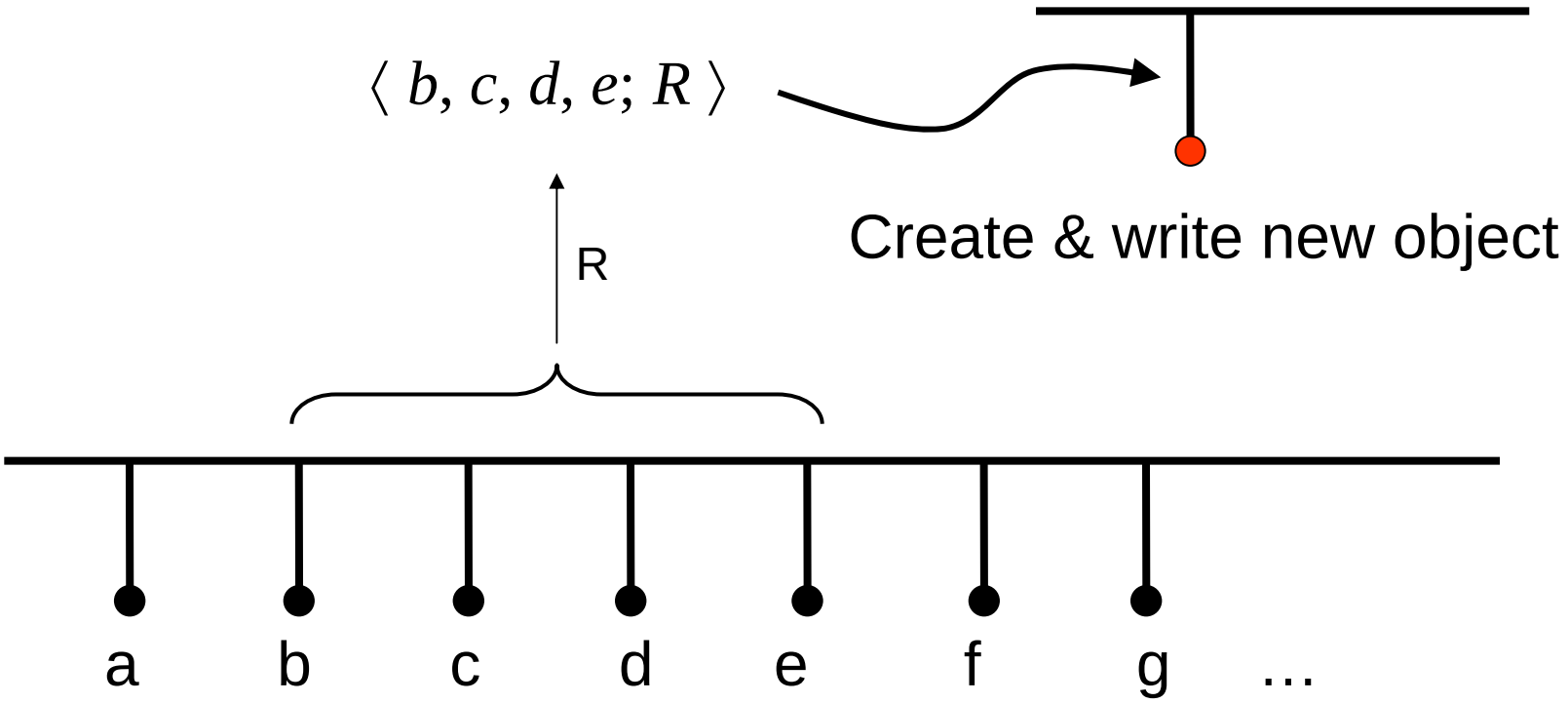


This is not compatible with orientation in simplicial complexes?

Simplicial complexes to hypernetworks

In chain hypernetworks, what do orientation & minus mean?



This is not compatible with orientation in simplicial complexes?

Conclusions

(1) Relational simplices are essential to represent systems

(2) … and to model systems of systems of systems

(3) The dynamics of n-ary assembly relations is fundamental

(4) Relational simplices are different to simplicial complexes

(5) There are some interesting questions $\in (SC)^C$