

Consensus and synchronization:
Cooperative dynamics on networks with delayed
and time-varying links

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Outline

- ▶ Network properties responsible for collective motion
- ▶ Abstract or generic models for dynamics - Realistic models for networks
- ▶ Time delays in information transmission and processing
- ▶ Time-varying connection structure
- ▶ Implications for social networks

Consensus problem

$$\dot{x}_i(t) = \varepsilon \sum_{j=1}^n a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \dots, n.$$

$x_i(t) \in \mathbb{R}$ is the state (“opinion”) of unit i at time t

$\varepsilon \geq 0$ is the coupling coefficient

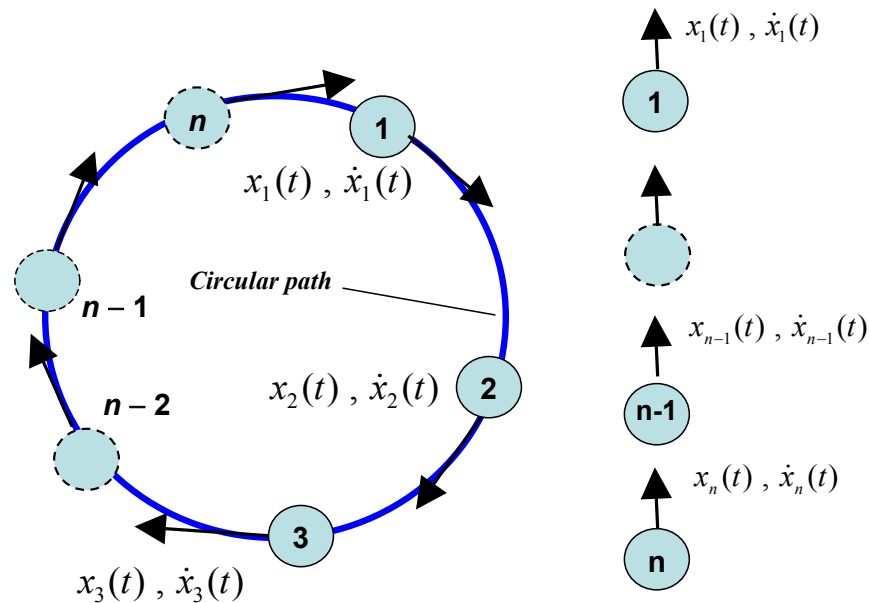
$a_{ij} \geq 0$ is the strength of the influence of unit j on i .

- ▶ The coupled system is said to reach consensus if for any set of initial conditions there exists some $c \in \mathbb{R}$ such that $\lim_{t \rightarrow \infty} x_i(t) = c$ for all i .
- ▶ The number c is then called the consensus value.

Traffic flow: Car following model

$$\dot{x}_i(t) = \varepsilon \sum_{j=1}^n a_{ij}(x_j(t) - x_i(t)), \quad x_i = \text{speed of vehicle } i.$$

- ▶ Sample spatial configurations for car following model:



- ▶ Desired behavior: All vehicles travel at a common speed regardless of variations in road and traffic conditions, etc.

Graph structure

- ▶ Connection topology: Adjacency matrix $A = [a_{ij}]$; defines a (weighted, directed) graph
- ▶ $d_i = \sum_j a_{ij} =$ in-degree at vertex i .
- ▶ Degree matrix $D = \text{diag}\{d_1, \dots, d_n\}$
- ▶ Laplacian $L = D - A$
- ▶ Laplacian eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. $\text{Re}(\lambda_i) \geq 0 \forall i$.
- ▶ $\lambda_1 = 0$ is always an eigenvalue; with eigenvector $(1, 1, \dots, 1)^\top$

Main consensus result

Coupled system in vector form, with $x = (x_1, \dots, x_n) \in \mathbb{R}^n$:

$$\dot{x} = -\varepsilon Lx(t)$$

- ▶ For $\varepsilon > 0$, consensus is reached if and only if

$$\min_{i \geq 2} \operatorname{Re}(\lambda_i) > 0.$$

- ▶ The spectral gap $\min_{i \geq 2} \operatorname{Re}(\lambda_i)$ provides a measure of the network's propensity for consensus. In general, it cannot be derived from the statistical properties of the network such as degree distribution, average distance, clustering coefficient etc.
(FA et al., IEEE Trans. Circuits and Systems (2006); Physica D (2006))

- ▶ Equivalent consensus condition in case of nonnegative weights:
Zero is a simple eigenvalue of L .
Geometrically: The graph has a spanning tree.
- ▶ The consensus value is $\langle u, x(0) \rangle$, where u is the left eigenvector of L corresponding to the zero eigenvalue.
- ▶ For undirected graphs, or for the directed ring configuration above,

$$u = \frac{1}{n}(1, 1, \dots, 1)$$

so the consensus value is the average of initial conditions.

- ▶ For the directed chain

$$u = (1, 0, \dots, 0)$$

so the car in front determines the common speed of the convoy.

Modeling driver reaction time

- ▶ $\tau =$ reaction delay

$$\dot{x}_i(t) = \sum_{j=1}^N a_{ij}(x_j(t - \tau) - x_i(t - \tau))$$

$$\dot{x}(t) = -\varepsilon Lx(t - \tau)$$

- ▶ More general delay model

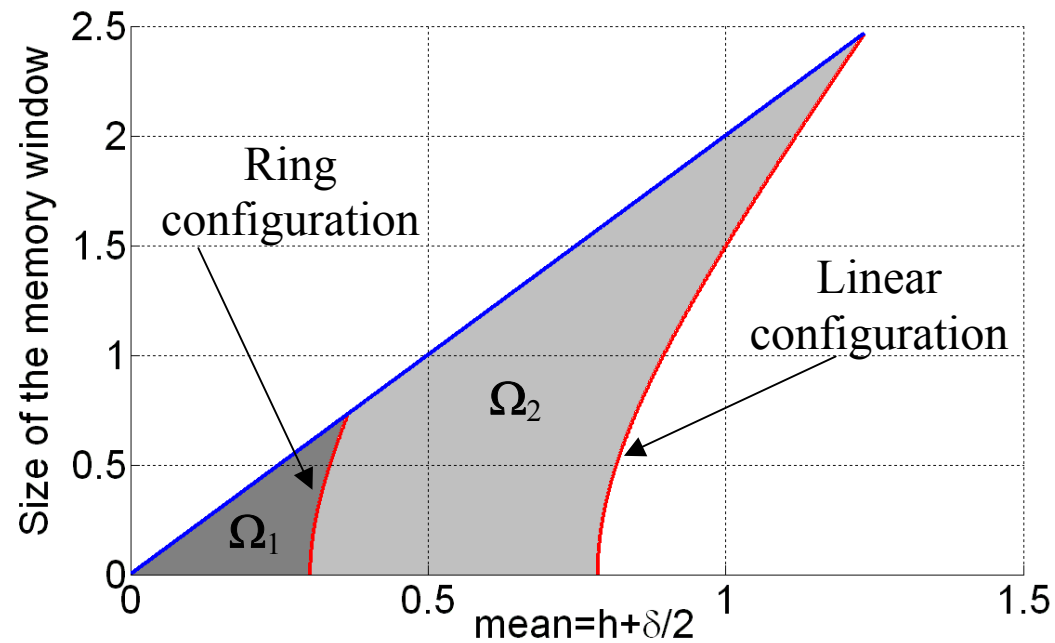
$$\dot{x}(t) = -\varepsilon L \int_0^\tau x(t - s) f(s) ds$$

which includes single or multiple delays as well as distributed delays. Distributed delays model short-term driver memory.

Stability with short-term driver memory

Uniform delay distribution with dead time modeling driver memory.

- Analytical stability regions



- ▶ Sufficiently large delays always destabilize synchronous traffic flow or consensus.
- ▶ Delay distributions improve stability.
(FA, Phys. Rev. Lett. 2003; Disc. Cont. Dyn. Syst. S 2008.)

Various forms of delays

- ▶ Undelayed consensus protocol

$$\dot{x}_i(t) = \varepsilon \sum_{j=1}^n a_{ij} (x_j(t) - x_i(t))$$

- ▶ Information processing delays

$$\dot{x}_i(t) = \varepsilon \sum_{j=1}^n a_{ij} (x_j(t - \tau) - x_i(t - \tau))$$

- ▶ Information propagation delays

$$\dot{x}_i(t) = \varepsilon \sum_{j=1}^n a_{ij} (x_j(t - \tau) - x_i(t))$$

Consensus with propagation delays

$$\dot{x}_i(t) = \frac{\varepsilon}{d_i} \sum_{j=1}^n a_{ij} \left(\int_0^\tau f(s) x_j(t-s) ds - x_i(t) \right)$$

$f(s) \geq 0$ for $s \in [0, \tau]$, and $\int_0^\tau f(s) ds = 1$.

- ▶ Vector form

$$\dot{x}(t) = -\varepsilon I x(t) + \varepsilon D^{-1} A \int_0^\tau f(s) x(t-s) ds.$$

- ▶ Normalized Laplacian $\bar{L} = I - D^{-1} A$

- ▶ Eigenvalues $\{\bar{\lambda}_1, \dots, \bar{\lambda}_n\}$. $\bar{\lambda}_1 = 0$; $\text{Re}(\bar{\lambda}_i) \geq 0 \forall i$.

Main consensus result

Theorem. Let $\varepsilon > 0$. The system reaches consensus if and only if zero is a simple eigenvalue of the Laplacian \bar{L} . Furthermore, for a fixed delay τ , the consensus value is

$$c = \frac{1}{1 + \varepsilon\tau} \left\langle u, x(0) + \varepsilon \int_{-\tau}^0 x(\xi) d\xi \right\rangle$$

and for a delay distribution given by f the consensus value is

$$c = \frac{1}{1 + \varepsilon\bar{\tau}} \left\langle u, x(0) + \varepsilon \int_0^{\tau} \int_{-\theta}^0 f(\theta)x(\xi) d\xi d\theta \right\rangle$$

where u is the left eigenvector of \bar{L} corresponding to the zero eigenvalue and $\bar{\tau} = \int_0^{\tau} sf(s) ds$ is the mean of the delay distribution f .

Time-varying networks

In many real networks, connections or the strength of connections change in time.

- ▶ Communication networks
- ▶ Networks of moving agents; obstacles between agents
- ▶ Failure and recovery of nodes in computer networks
- ▶ Social networks

Consensus in time-varying networks

$$x_i(t + 1) = x_i(t) + \varepsilon \sum_{j=1}^n a_{ij}(t)(x_j(t) - x_i(t))$$

- ▶ Let $G = I - \varepsilon L$, and $x = (x_1, \dots, x_n)$. Then dynamics become

$$x(t + 1) = G(t)x(t).$$

- ▶ $G(t)$ is a stochastic matrix if its diagonal elements are nonnegative. (i.e. a nonnegative matrix with row sums equal to 1, provided ε is not too large.)

Synchronization problems

$$x_i(t + 1) = f(x_i(t)) + \varepsilon \sum_{j=1}^n a_{ij}(t)(f(x_j(t)) - f(x_i(t)))$$

f is some nonlinear function.

- ▶ The system is said to synchronize if $|x_i(t) - x_j(t)| \rightarrow 0 \forall i, j$ as $t \rightarrow \infty$.

- ▶ In vector notation

$$x(t + 1) = G(t)F(x(t))$$

where $F(x) = (f(x_1), \dots, f(x_n))$.

- ▶ Consensus is a special case with f being the identity function.

Some mathematical ideas

- ▶ If G is a matrix with row vectors g_1, \dots, g_n , its *Hajnal diameter* is defined as

$$\text{diam}(G) = \max_{i,j} \|g_i - g_j\|$$

- ▶ Let Γ denote the infinite matrix sequence $G(0), G(1), \dots$. We define its diameter

$$\text{diam}(\Gamma) = \limsup_{t \rightarrow \infty} \left\{ \text{diam} \left[\prod_{k=0}^{t-1} G(k) \right] \right\}^{1/t}$$

- ▶ Note the similarity to the definition of the *Lyapunov exponent* μ of the function f :

$$\mu = \limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^{t-1} \log |f'(x(k))|$$

Main results

Lemma. Log of the diameter equals the largest transverse Lyapunov exponent.

Theorem. The network synchronizes if

$$\log \text{diam}(\Gamma) + \mu < 0$$

where μ is the Lyapunov exponent of the function f .

- ▶ $\text{diam}(\Gamma)$ measures the synchronizability of a time-varying network.

Theorem. $\log \text{diam}(\Gamma) < 0$ iff $\exists T \in \mathbb{Z}^+$ such that the union of the graphs over any time interval of length T contains a *spanning tree*.

- ▶ Possible to have synchronization even though the network may be disconnected at every instant of time.

Example: Blinking scale-free networks.

- ▶ The model evolves with malfunction and recovery. Each node is a chaotic logistic function.
- ▶ At time $t = 0$, the initial graph is a scale-free network of 500 nodes and average degree 12.
- ▶ At each time $t \geq 1$, every vertex i malfunctions with probability p . If a vertex malfunctions, all edges linked to it disappear.
- ▶ In addition, a malfunctioned vertex recovers after a time interval T (here 3) and then causes the reestablishment of all edges linked to it in the initial graph Γ .

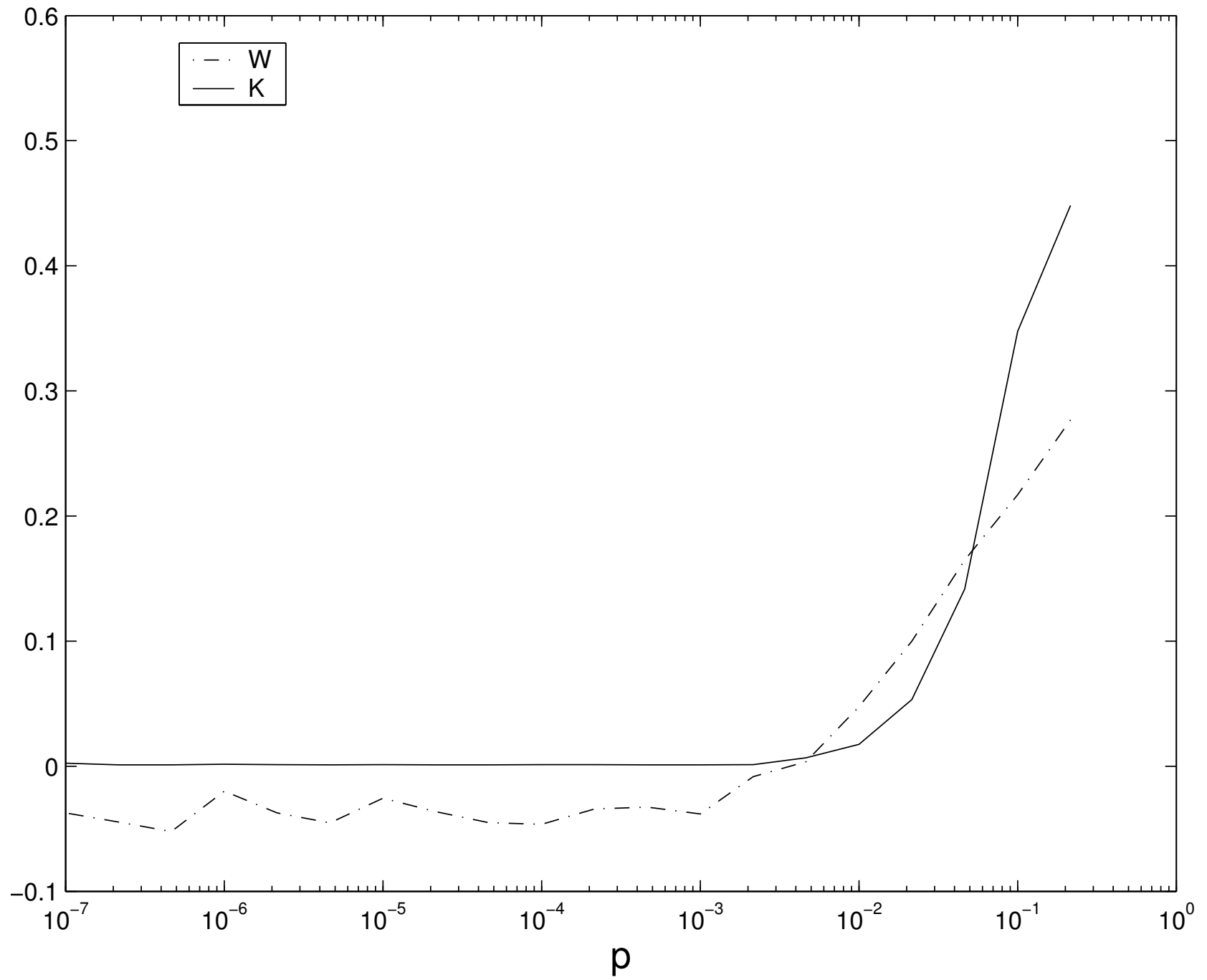
- ▶ We measure synchronization by the time-averaged variance

$$K = \frac{1}{n-1} \sum_{i=1}^n \langle (x_i - \bar{x})^2 \rangle$$

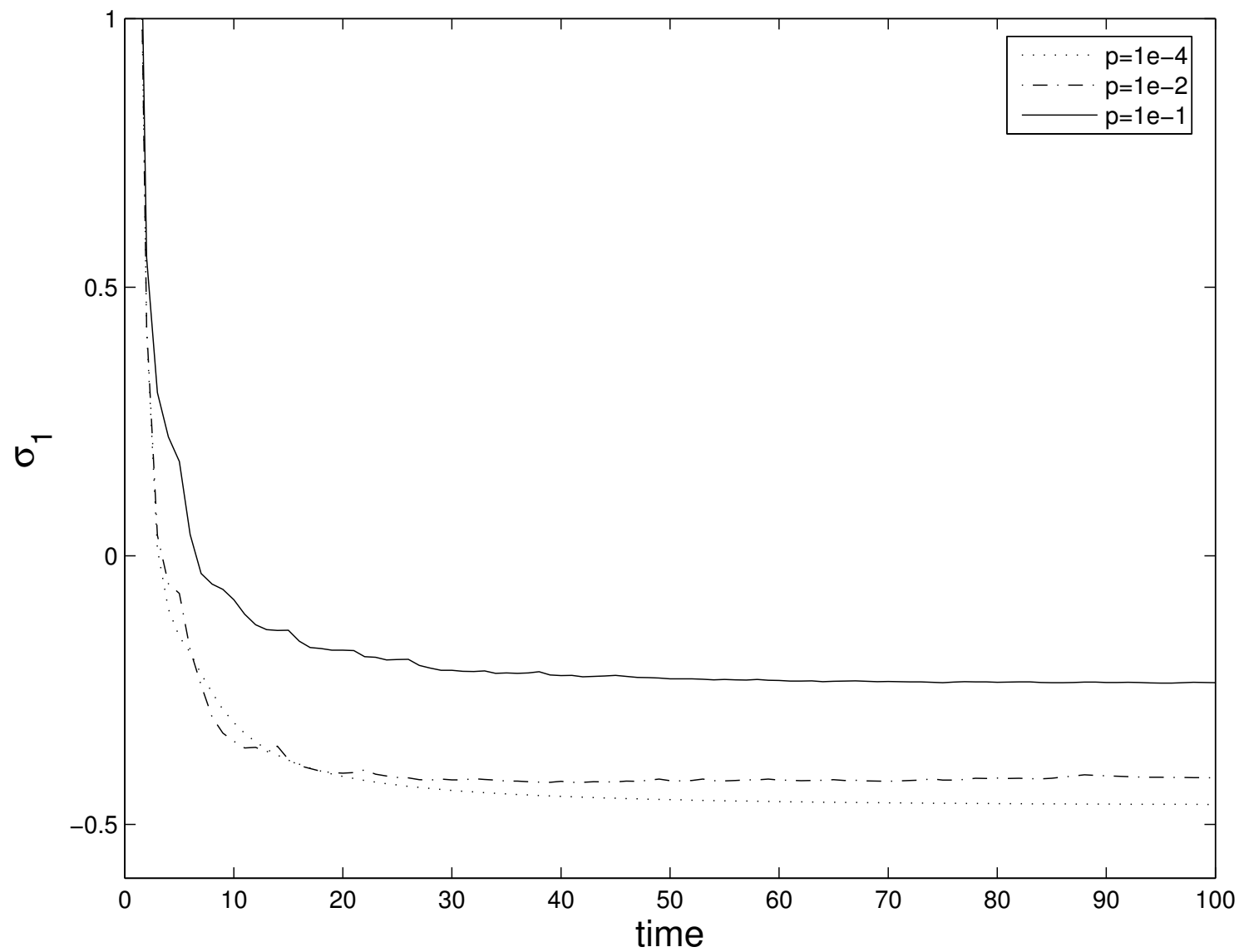
and denote

$$W = \log \text{diam}(\Gamma) + \mu.$$

- ▶ Denote σ_1 as the largest Lyapunov exponent transverse to the synchronization direction. Then σ_1 is equivalent to W .



Synchronization speed



Summary and outlook

- ▶ Synchronization properties of delayed networks depend on the interplay between the spectrum of the coupling operator and the time delays.
- ▶ Synchronization of time-varying networks is characterized by the Hajnal diameter of the associated graph sequence.
- ▶ Delays and time-varying connections can improve synchronizability of the network, but there are also instances where they impede synchronization.
- ▶ Networks with both delays and time-varying links. System can synchronize to a periodic trajectory for certain delays.
Wu, FA, Jost, *Networks & Heterogeneous Media* (2011).
- ▶ Symbolic dynamics approach to delayed time-varying networks.
FA et al., *Phys. Lett. A* (2010).